# Image Processing

## Processing in the Frequency Domain

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Class 05: Processing in the Frequency Domain

# Fourier Series

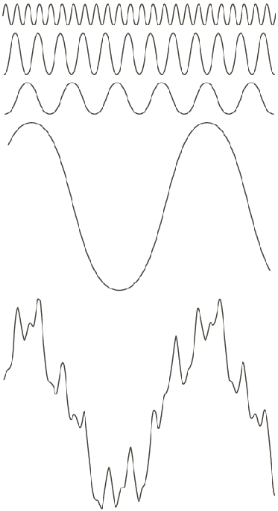


**FIGURE 4.1** The function at the bottom is the sum of the four functions above it. Fourier's idea in 1807 that periodic functions could be represented as a weighted sum of sines and cosines was met with skepticism.

# Fourier Series

## Fourier Series

Analysis

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi nt}{T}}$$

Synthesis

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-j\frac{2\pi nt}{T}} dt$$

# 1D Fourier Transformada

## Continuous Fourier Transform

Analysis

$$F\left\{f(t)\right\} = F(\mu) = \int_{-\infty}^{\infty} f(t)e^{-j2\pi\mu t}dt$$

Synthesis

$$f(t) = \int_{-\infty}^{\infty} F(\mu)e^{j2\pi\mu t}d\mu$$

# 1D Continuous Fourier Transformada

Ex 1: Calculate the FT of $f(t)$

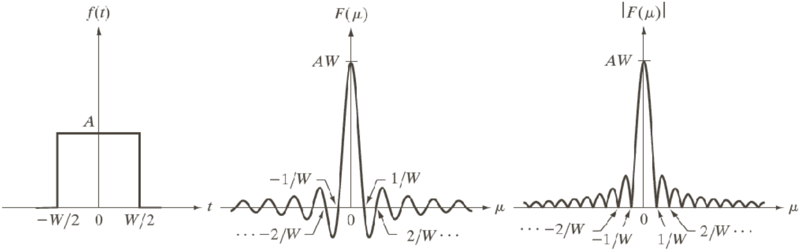
$$f(t) = \begin{cases} 0 & \text{if } |t| > W/2 \\ A & \text{if } |t| \leq W/2 \end{cases}$$

$$\begin{aligned} F(\mu) &= \int_{-\infty}^{\infty} f(t) e^{-j2\pi\mu t} dt = \int_{-W/2}^{W/2} A e^{-j2\pi\mu t} dt \\ &= \frac{-A}{j2\pi\mu} \left[ e^{-j2\pi\mu t} \right]_{W/2}^{W/2} = \frac{-A}{j2\pi\mu} \left[ e^{-j\pi\mu W} - e^{j\pi\mu W} \right] \\ &= AW \frac{\sin(\pi\mu W)}{\pi\mu W} = AW \mathrm{sinc}(\mu W) \end{aligned}$$

Ex 1: Calculate the FT of $f(t)$

$$f(t) = \begin{cases} 0 & \text{if } |t| > W/2 \\ A & \text{if } |t| \leq W/2 \end{cases}$$

$$F(\mu) = AW\text{sinc}(\mu W)$$




a b c

**FIGURE 4.4** (a) A simple function; (b) its Fourier transform; and (c) the spectrum. All functions extend to infinity in both directions.

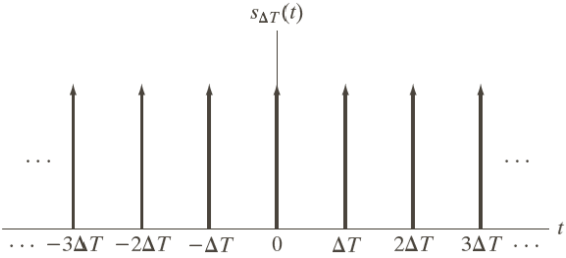
Ex 2: Calculate the FT of $f(t) = \delta(t)$

$$
\begin{aligned}
F(\mu) &= \int_{-\infty}^{\infty} f(t) e^{-j2\pi\mu t} dt \\
&= \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi\mu t} dt = e^{-j2\pi\mu 0} = 1
\end{aligned}
$$

Ex 3: Calculate the FT of $f(t) = \delta(t - t_0)$

$$
\begin{aligned}
F(\mu) &= \int_{-\infty}^{\infty} f(t) e^{-j2\pi\mu t} dt \\
&= \int_{-\infty}^{\infty} \delta(t - t_0) e^{-j2\pi\mu t} dt = e^{-j2\pi\mu t_0} \\
&= \cos(2\pi\mu t_0) - j \sin(2\pi\mu t_0)
\end{aligned}
$$

# 1D Continuous Fourier Transformada




Ex 4: Calculate the FT of a *train of impulses*

$$s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta T)$$

$$F\{f(t)\} = F(\mu) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi\mu t} dt$$

$$f(t) = \int_{-\infty}^{\infty} F(\mu) e^{j2\pi\mu t} d\mu$$

## Symmetry Property

$$f(t) \rightarrow F(\mu)$$

$$F(t) \rightarrow f(-\mu)$$

# 1D Continuous Fourier Transformada

$$F\{f(t)\} = F(\mu) = \int_{-\infty}^{\infty} f(t)e^{-j2\pi\mu t}dt$$

$$f(t) = \int_{-\infty}^{\infty} F(\mu)e^{j2\pi\mu t}d\mu$$

## Symmetry Property

$$f(t) \rightarrow F(\mu)$$

$$F(t) \rightarrow f(-\mu)$$

Given that:

$$\delta(t - t_0) \rightarrow e^{-j2\pi t_0 t}$$

we have:

$$e^{-j2\pi t_0 t} \rightarrow \delta(-\mu - t_0)$$

making $-t_0 = a$

$$e^{j2\pi at} \rightarrow \delta(-\mu + a) = \delta(\mu - a)$$

# 1D Continuous Fourier Transformada

Ex 4: Given that $s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta T)$ is a periodic signal, we can express the train of impulses using a Fourier Series:

$$s_{\Delta T} = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{\Delta T}t}$$

where

$$\begin{aligned} c_n &= \frac{1}{\Delta T}\int_{-\Delta T/2}^{\Delta T/2} s_{\Delta T}(t) e^{-j\frac{\pi nt}{\Delta T}} dt \\ &= \frac{1}{\Delta T}\int_{-\Delta T/2}^{\Delta T/2} \delta(t) e^{-j\frac{\pi nt}{\Delta T}} dt \\ &= \frac{1}{\Delta T} e^{O} = \frac{1}{\Delta T} \end{aligned}$$

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$$s_{\Delta T} = \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi n}{\Delta T}t}$$

Ex 4:

$$F\left\{s_{\Delta T}\right\} = F\left\{\frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi n}{\Delta T}t}\right\}$$

$$F\left\{e^{j\frac{2\pi n}{\Delta T}t}\right\} = \delta\left(\mu - \frac{n}{\Delta T}\right)$$

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$$F\left\{e^{j\frac{2\pi n}{\Delta T}t}\right\} = \delta\left(\mu - \frac{n}{\Delta T}\right)$$

$$F\left\{s_{\Delta T}\right\} = S(\mu) = F\left\{\frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi n}{\Delta T}t}\right\} = \frac{1}{\Delta T}F\left\{\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi n}{\Delta T}t}\right\}$$
$$= \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty}\delta\left(\mu - \frac{n}{\Delta T}\right)$$

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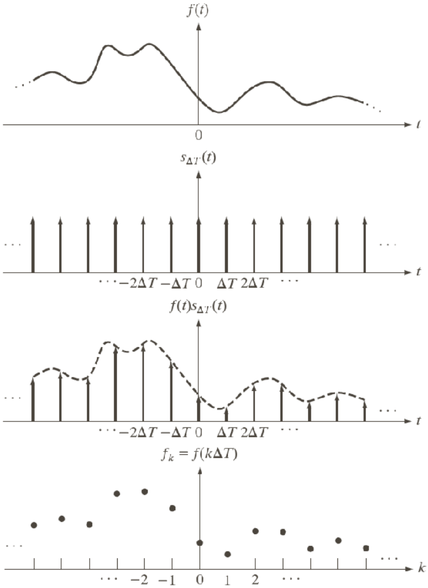
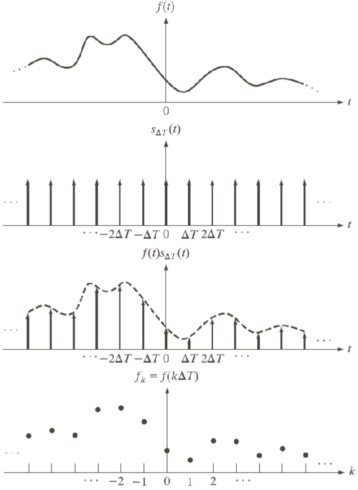
$$= \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} \delta\left(\mu - \frac{n}{\Delta T}\right)$$

Another train of impulses!

# Sampling


$f(t)$
0
$t$
$s_{\Delta T}(t)$
$\cdots -2\Delta T\ -\Delta T\ 0\ \ \Delta T\ 2\Delta T\ \cdots$
$t$
$f(t)s_{\Delta T}(t)$
$\cdots -2\Delta T\ -\Delta T\ 0\ \ \Delta T\ 2\Delta T\ \cdots$
$t$
$f_k = f(k\Delta T)$
$\cdots\ -2\ -1\ 0\ 1\ 2\ \cdots$
$k$




Sampled signal:

$$\tilde{f}(t) = f(t) \cdot s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} f(t)\delta(t - n\Delta T)$$

$$f_k = \int_{-\infty}^{\infty} f(t)\delta(t - k\Delta T)dt = f(k\Delta T)$$

Taking the Fourier Transform of the sampled signal:

$$\tilde{f}(t) = f(t) \cdot s_{\Delta T}(t)$$

$$f_k = \int_{-\infty}^{\infty} f(t)\delta(t - k\Delta T)dt$$

$$\begin{aligned} F\left\{\tilde{f}(t)\right\} = \tilde{F}(\mu) &= F\left\{f(t) \cdot s_{\Delta T}(t)\right\} \\ &= F(\mu) * S(\mu) \end{aligned}$$

Taking the Fourier Transform of the sampled signal:

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One more property: The FT of the convolution

# Sampling

$$f(t) * h(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau$$

$$F\{f(t) * h(t)\} = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau\right] e^{-j2\pi\mu t}dt$$

$$= \int_{-\infty}^{\infty} f(\tau) \left[\int_{-\infty}^{\infty} h(t-\tau)e^{-j2\pi\mu t}dt\right] d\tau$$

# Sampling

$$f(t) * h(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau$$

$$F\{f(t) * h(t)\} = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau\right] e^{-j2\pi\mu t}dt$$

$$= \int_{-\infty}^{\infty} f(\tau)\left[\int_{-\infty}^{\infty} h(t-\tau)e^{-j2\pi\mu t}dt\right] d\tau$$

$$F\{h(t)\} = H(\mu)$$

$$F\{h(t-t_0)\} = \int_{-\infty}^{\infty} h(t-\tau)e^{-j2\pi\mu t}dt = \int_{-\infty}^{\infty} h(t_1)e^{-j2\pi\mu(t_1+\tau)}dt_1$$

$$= e^{-j2\pi\mu\tau}\int_{-\infty}^{\infty} h(t_1)e^{-j2\pi\mu t_1)}dt_1$$

$$= H(\mu)e^{-j2\pi\mu\tau}$$

# Sampling

$$F\{f(t) * h(t)\} = \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} f(\tau) h(t-\tau) d\tau \right] e^{-j2\pi\mu t} dt$$
$$= \int_{-\infty}^{\infty} f(\tau) \left[ \int_{-\infty}^{\infty} h(t-\tau) e^{-j2\pi u t} dt \right] d\tau$$

$$F\{h(t-t_0)\} = H(\mu) e^{-j2\pi\mu\tau}$$

$$F\{f(t) * h(t)\} = \int_{-\infty}^{\infty} f(\tau) \left[ H(\mu) e^{-j2\pi\mu\tau} \right] d\tau$$
$$= H(\mu) \int_{-\infty}^{\infty} f(\tau) e^{-j2\pi\mu\tau} d\tau = H(\mu) F(\mu)$$

# Properties of the Fourier Transform

## Convolution

$$f(t) * h(t) \rightarrow F(\mu)H(\mu)$$

## Modulation

$$f(t) \cdot h(t) \rightarrow F(\mu) * H(\mu)$$

# Sampling

$$\tilde{F}(\mu) = F\left\{\tilde{f}(t)\right\} = \tilde{F}(\mu) = F\left\{f(t) \cdot s_{\Delta T}(t)\right\}$$
$$= F(\mu) * F\left\{s_{\Delta T}(t)\right\} = F(\mu) * S(\mu)$$

# Sampling

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$$S(\mu) = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} \delta\left(\mu - \frac{n}{\Delta T}\right)$$

# Sampling

$$\tilde{F}(\mu) = F\left\{\tilde{f}(t)\right\} = \tilde{F}(\mu) = F\left\{f(t)\cdot s_{\Delta T}(t)\right\}$$
$$= F(\mu) * F\left\{s_{\Delta T}(t)\right\} = F(\mu) * S(\mu)$$

$$S(\mu) = \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty}\delta\left(\mu - \frac{n}{\Delta T}\right)$$

$$\tilde{F}(\mu) = F(\mu) * S(\mu) = \int_{-\infty}^{\infty} F(\tau)S(\mu-\tau)d\tau$$
$$= \frac{1}{\Delta T}\int_{-\infty}^{\infty} F(\tau)\sum_{n=-\infty}^{\infty}\delta\left(\mu - \tau - \frac{n}{\Delta T}\right)d\tau$$

# Sampling

$$
\begin{aligned}
\tilde{F}(\mu) &= F(\mu) * S(\mu) = \int_{-\infty}^{\infty} F(\tau) S(\mu - \tau) d\tau \\
&= \frac{1}{\Delta T} \int_{-\infty}^{\infty} F(\tau) \sum_{n=-\infty}^{\infty} \delta \left( \mu - \tau - \frac{n}{\Delta T} \right) d\tau \\
&= \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} F(\tau) \delta \left( \mu - \tau - \frac{n}{\Delta T} \right) d\tau \\
&= \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} F \left( \mu - \frac{n}{\Delta T} \right)
\end{aligned}
$$

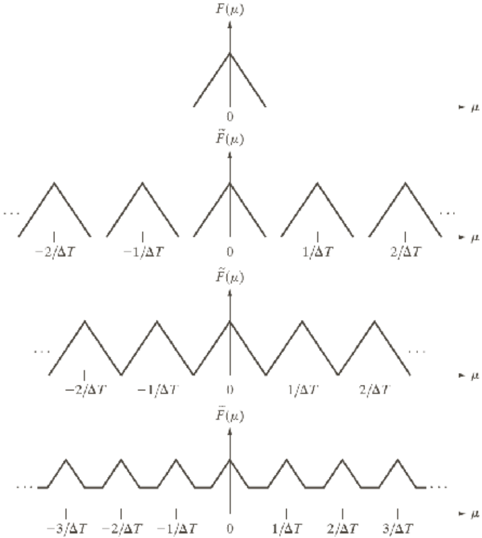
# Sampling

$$\begin{aligned}\tilde{F}(\mu) = F(\mu) * S(\mu) &= \int_{-\infty}^{\infty} F(\tau) S(\mu - \tau) d\tau \\ &= \frac{1}{\Delta T} \int_{-\infty}^{\infty} F(\tau) \sum_{n=-\infty}^{\infty} \delta\left(\mu - \tau - \frac{n}{\Delta T}\right) d\tau \\ &= \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} F(\tau) \delta\left(\mu - \tau - \frac{n}{\Delta T}\right) d\tau \\ &= \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} F\left(\mu - \frac{n}{\Delta T}\right)\end{aligned}$$

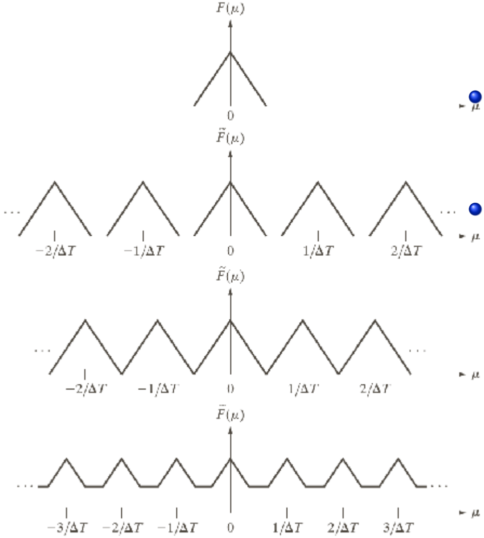
The FT $\tilde{F}(\mu)$ of the sampled signal $\tilde{f}(t)$:

- sequence (peridical and infinite) of shifted copies of $F(\mu)$;
- the separation between the shifted copies is $\frac{1}{\Delta T}$;
- continuous function ($F(\mu)$).

# Sampling

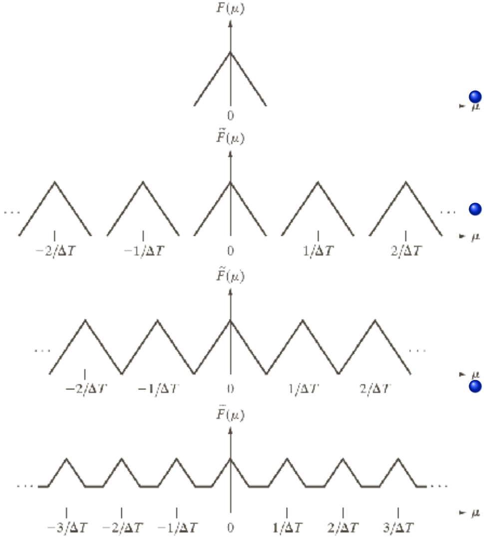


$F(\mu)$
0
$\mu$
$\tilde{F}(\mu)$
. . .
$-2/\Delta T$
$-1/\Delta T$
0
$1/\Delta T$
$2/\Delta T$
. . .
$\mu$
$\tilde{F}(\mu)$
. . .
$-2/\Delta T$
$-1/\Delta T$
0
$1/\Delta T$
$2/\Delta T$
. . .
$\mu$
$\tilde{F}(\mu)$
. . .
$-3/\Delta T$
$-2/\Delta T$
$-1/\Delta T$
0
$1/\Delta T$
$2/\Delta T$
$3/\Delta T$
. . .
$\mu$




- $\frac{1}{\Delta T}$ is the sampling rate used to generate the discrete signal;
- From the spectrum, what value of $\frac{1}{\Delta T}$:
  - big enough;
  - sufficient;
  - too small.

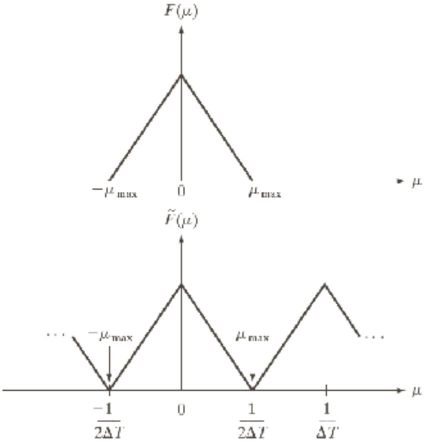
# Sampling



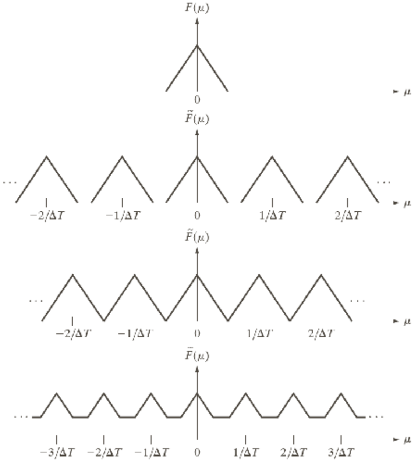

- $\frac{1}{\Delta T}$ is the sampling rate used to generate the discrete signal;
- From the spectrum, what value of $\frac{1}{\Delta T}$:
  - big enough;
  - sufficient;
  - too small.
- What is the minimum distance?

# Sampling Theorem

Assuming that a signal $f(t)$ have a Fourier Transform with zero values outside the interval $[-\mu_{max}, \mu_{max}]$, i.e., a signal with a limited bandwidth.

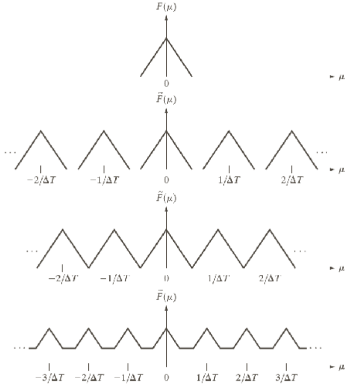


What conditions must be satisfied for the sinal to be recovered without major problems? (without the superposition of the signal)



$$\frac{1}{\Delta T} > 2\mu_{max} = f_s$$

Nyquist rate – Sampling Theorem



How to recover the signal?

How to recover the signal? Consider a signal $h(t)$, whose Fourier transform is given by:

$$H(\mu) = \begin{cases} \Delta T & \text{if} -u_{max} \leq \mu \leq \mu_{max} \\ 0 & \text{otherwise} \end{cases}$$

When we multiply a periodic sequence by the function:

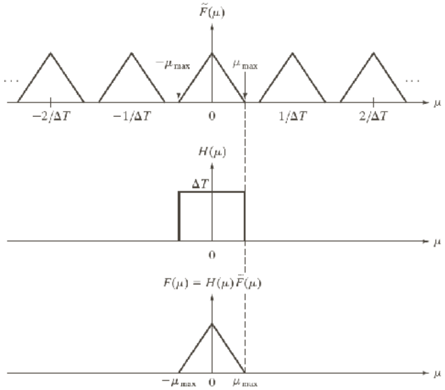
$$F(\mu) = H(\mu)\tilde{F}(\mu)$$

Then, we obtain $f(t)$ by taking the inverse Fourier Transform of $F(\mu)$

$$f(t) = \int_{-\infty}^{\infty} F(\mu)e^{j2\pi\mu t}d\mu$$

Low-Pass Filter:

$$H(\mu) = \begin{cases} \Delta T & \text{if} -u_{max} \leq \mu \leq \mu_{max} \\ 0 & \text{otherwise} \end{cases}$$



# Sampling Theorem

What happens when the Nyquist rate is not respected?

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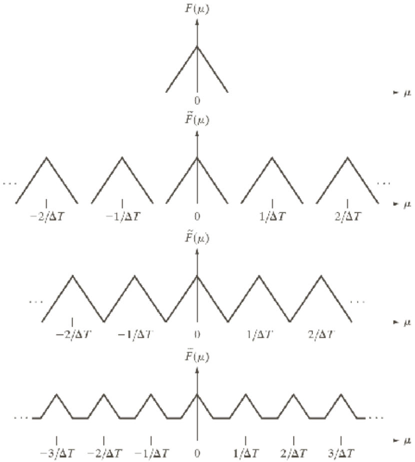
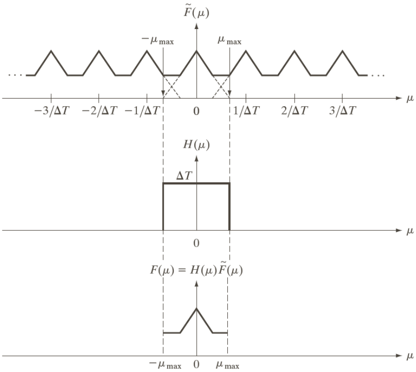
What happens when the Nyquist rate is not respected? ALIAS!



What happens when the Nyquist rate is not respected? ALIAS!

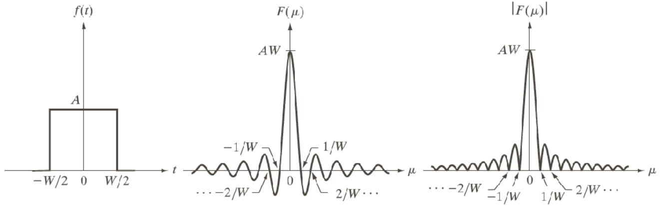


# Sampling Theorem

- Alias cannot be avoided ... Even if the signal is limited in band ...

$$h_p(t) = \begin{cases} 1 & \text{if} 0 \leq t \leq T \\ 0 & \text{otherwise} \end{cases}$$

- The convolution of $h_p(t)$ and $f(t)$ generates a signal with infinite frequencies;




a b c

**FIGURE 4.4** (a) A simple function; (b) its Fourier transform; and (c) the spectrum. All functions extend to infinity in both directions.

# Sampling Theorem

$$
\begin{aligned}
f(t) &= \int_{-\infty}^{\infty} F(\mu) e^{j2\pi\mu t} d\mu \\
&= \int_{-\infty}^{\infty} \left( H(\mu)\tilde{F}(\mu) \right) e^{j2\pi\mu t} d\mu \\
&= h(t) * \tilde{f}(t)
\end{aligned}
$$

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&= h(t) * \tilde{f}(t)
\end{aligned}
$$

as

$$
\tilde{f}(t) = f(t) s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} f(t)\delta(t - n\Delta T)
$$

# Sampling Theorem

$$f(t) = \int_{-\infty}^{\infty} F(\mu) e^{j2\pi\mu t} d\mu$$
$$= \int_{-\infty}^{\infty} \left( H(\mu)\tilde{F}(\mu) \right) e^{j2\pi\mu t} d\mu$$
$$= h(t) * \tilde{f}(t)$$

as

$$\tilde{f}(t) = f(t) s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} f(t)\delta(t - n\Delta T)$$

$$f(t) = \sum_{-\infty}^{\infty} f(n\Delta T) \text{sinc}\left[(t - n\Delta T)/n\Delta T\right]$$

# Discrete Fourier Transform

- A TF de um sinal de um sinal amostrado e limitado em banda, se extende no tempo de $-\infty$ a $\infty$ e é uma função *contínua* e *periódica*;

# Discrete Fourier Transform

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- E uma transformada adequada.

# Discrete Fourier Transform

- A TF de um sinal de um sinal amostrado e limitado em banda, se extende no tempo de $-\infty$ a $\infty$ e é uma função *contínua* e *periódica*;
- Na prática, devemos ter um número finito de amostras;
- E uma transformada adequada.

$$
\begin{aligned}
\tilde{F}(\mu) &= \int_{-\infty}^{\infty} \tilde{f}(t) e^{-j2\pi\mu t} d\mu \\
&= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f(t)\delta(t - n\Delta T) e^{-j2\pi\mu t} d\mu \\
&= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)\delta(t - n\Delta T) e^{-j2\pi\mu t} d\mu \\
&= \sum_{n=-\infty}^{\infty} f_n e^{-j2\pi\mu n\Delta T}
\end{aligned}
$$

# Discrete Fourier Transform

- Como $\tilde{F}(\mu)$ é contínuo e periódico com período $1/\Delta T$
- Logo, para caracterização do sinal precisamos apenas de 1 período amostrado de $\tilde{F}(\mu)$.

# Discrete Fourier Transform

- Como $\tilde{F}(\mu)$ é contínuo e periódico com período $1/\Delta T$
- Logo, para caracterização do sinal precisamos apenas de 1 período amostrado de $\tilde{F}(\mu)$.
- Suponha que em um período $(0 \leq \mu \leq 1/\Delta T)$ de $\tilde{F}(\mu)$ queremos coletar $M$ amostras:

$$\mu = \frac{m}{M\Delta T}, \quad m = 0, 1, 2, \ldots, M-1.$$

# Discrete Fourier Transform

- Como $\tilde{F}(\mu)$ é contínuo e periódico com período $1/\Delta T$
- Logo, para caracterização do sinal precisamos apenas de 1 período amostrado de $\tilde{F}(\mu)$.
- Suponha que em um período $(0 \leq \mu \leq 1/\Delta T)$ de $\tilde{F}(\mu)$ queremos coletar $M$ amostras:

$$\mu = \frac{m}{M\Delta T}, \qquad m = 0, 1, 2, \ldots, M-1.$$

$$\tilde{F}(\mu) = \sum_{n=-\infty}^{\infty} f_n e^{-j2\pi\mu n\Delta T}$$

$$F_m = \sum_{n=0}^{M-1} f_n e^{-j2\pi mn/M} \quad m = 0, 1, 2, \ldots, M-1.$$

Esta é expressão da TF discreta que estamos procurando.

# Discrete Fourier Transform

## Discrete Fourier Transform (TDF) - Direct

$$F(u) = \sum_{x=0}^{M-1} f(x) e^{-j2\pi ux/M} \quad m = 0, 1, 2, \ldots, M-1.$$

## Discrete Fourier Transform (TDF) - Inverse

$$f(x) = \frac{1}{M} \sum_{u=0}^{M-1} F(u) e^{j2\pi ux/M} \quad x = 0, 1, 2, \ldots, M-1.$$

# Discrete Fourier Transform

$$F(u) = F(u + kM)$$

$$f(x) = f(x + kM)$$

# Discrete Fourier Transform

$$F(u) = F(u + kM)$$

$$f(x) = f(x + kM)$$

And the convolution? Circular Convolution

$$f(x) * h(x) = \sum_{m=0}^{M-1} f(m)h(x - m)$$

# Discrete Fourier Transform

$$F(u) = F(u + kM)$$

$$f(x) = f(x + kM)$$

And the convolution? Circular Convolution

$$f(x) * h(x) = \sum_{m=0}^{M-1} f(m)h(x - m)$$

Relation between spatial sampling and frequency intervals

$$T = M\Delta T$$

$$\Delta u = 1/(M\Delta T) = 1/T$$

$$\Omega = M\Delta u = 1/\Delta T$$

# Discrete Fourier Transform

$$X[k] = \begin{cases} \sum_{n=0}^{N-1} x[n] W_N^{kn}, & 0 \leq k \leq N-1 \\ 0, & \text{c.c.} \end{cases},$$

$$x[n] = \begin{cases} 1/N \sum_{k=0}^{N-1} X[k] W_N^{-kn}, & 0 \leq n \leq N-1 \\ 0, & \text{c.c.} \end{cases}.$$

A partir destas relações que definimos a Transformada Discreta de Fourier (TDF) de N amostras.

**Análise:**

$$\boxed{X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}} \quad 0 \leq k \leq N-1, \tag{18}$$

$$W_N = e^{-j2\pi n/N}.$$

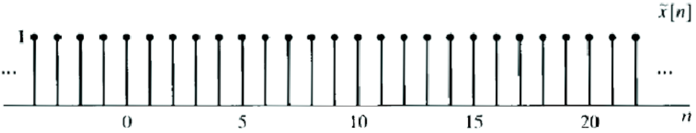
**Síntese:**

$$\boxed{x[n] = 1/N \sum_{k=0}^{N-1} X[k] W_N^{-kn}}, \quad 0 \leq n \leq N-1. \tag{19}$$

# Example

DFT of a step signal of length 5:

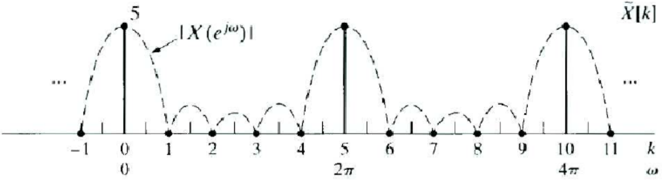
(a)



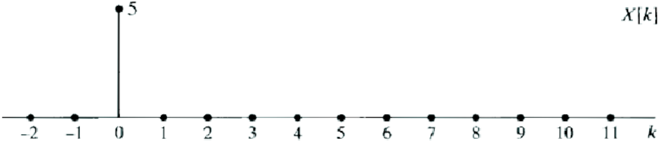
(b)

DFT of a step signal of length 5:

$$\tilde{X}[k] = \sum_{n=0}^{4} 1e^{-j(2\pi/5)\cdot kn} = 5\delta[k - r\cdot 5]$$



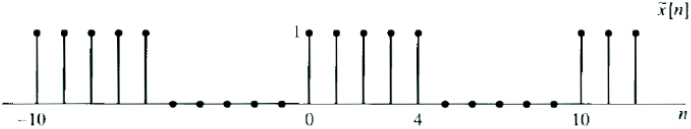

(c)



# Example

DFT of a step signal of length 10:


(a)

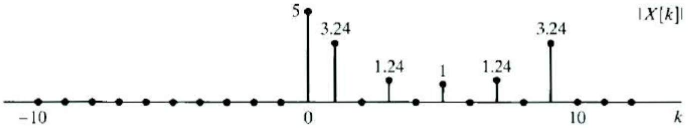



(b)

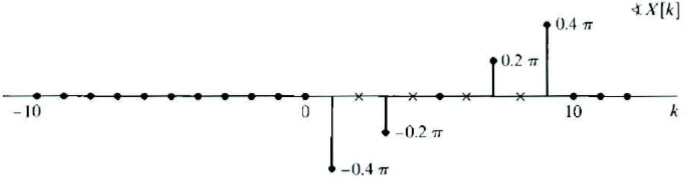
DFT of a step signal of length 10:

$$\tilde{X}[k] = \sum_{n=0}^{4} 1 e^{-j(2\pi/10)\cdot kn} = \frac{1 - e^{-j(2\pi)k}}{1 - e^{-j(2\pi/5)k}} = e^{-j(4\pi/10)k} \frac{\sin(\pi k/2)}{\sin(\pi k/10)}$$

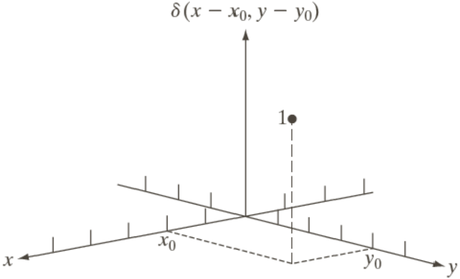
(b)

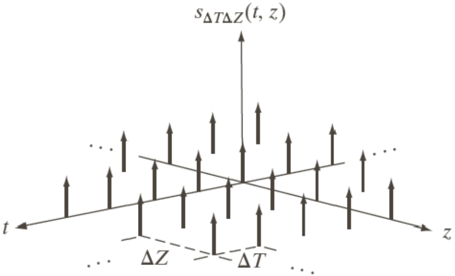



(c)



# 2D Signals



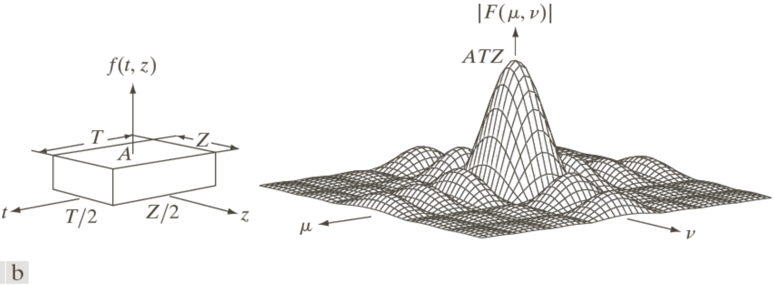
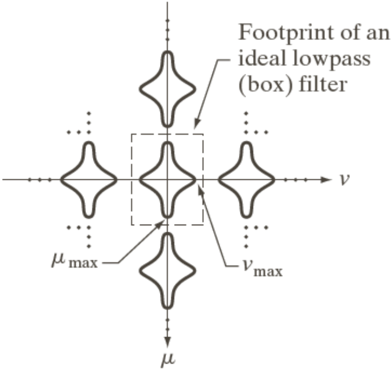
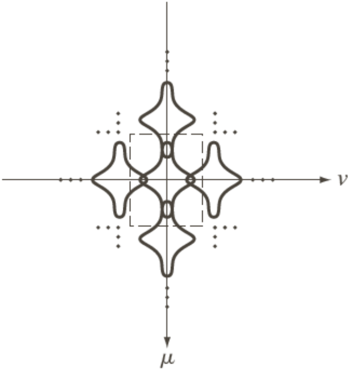



**FIGURE 4.13** (a) A 2-D function, and (b) a section of its spectrum (not to scale). The block is longer along the $t$-axis, so the spectrum is more "contracted" along the $\mu$-axis. Compare with Fig. 4.4.


Footprint of an ideal lowpass (box) filter
v
μ
μ max
v max


v
μ

# Discrete Fourier Transform

$$F(u,v) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j(2\pi)\cdot\left(\frac{ux}{M}+\frac{vy}{N}\right)}$$

# Discrete Fourier Transform

$$F(u,v) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) e^{-j(2\pi)\cdot\left(\frac{ux}{M}+\frac{vy}{N}\right)}$$

$$f(x,y) = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u,v) e^{j(2\pi)\cdot\left(\frac{ux}{M}+\frac{vy}{N}\right)}$$

# Discrete Fourier Transform

$$F(u,v) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j(2\pi)\cdot\left(\frac{ux}{M}+\frac{vy}{N}\right)}$$

$$f(x,y) = \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)e^{j(2\pi)\cdot\left(\frac{ux}{M}+\frac{vy}{N}\right)}$$

Magnitude:

$$|F(u,v)| = \left[R^2(u,v) + I^2(u,v)\right]^{1/2}$$

Fase:

$$|\phi(u,v)| = \arctan\left[\frac{I(u,v)}{R(u,v)}\right]$$

# Properties

| | |
|---|---|
| Fourier transform | $F(u,v) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux/M+vy/N)}$ |
| Inverse Fourier transform | $f(x,y) = \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)e^{j2\pi(ux/M+vy/N)}$ |
| Polar representation | $F(u,v) = \lvert F(u,v)\rvert e^{-j\phi(u,v)}$ |
| Spectrum | $\lvert F(u,v)\rvert = \left[R^2(u,v) + I^2(u,v)\right]^{1/2}$, $R = \text{Real}(F)$ and $I = \text{Imag}(F)$ |
| Phase angle | $\phi(u,v) = \tan^{-1}\left[\frac{I(u,v)}{R(u,v)}\right]$ |
| Power spectrum | $P(u,v) = \lvert F(u,v)\rvert^2$ |
| Average value | $\bar{f}(x,y) = F(0,0) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)$ |
| Translation | $f(x,y)e^{j2\pi(u_0x/M+v_0y/N)} \Leftrightarrow F(u-u_0, v-v_0)$<br>$f(x-x_0, y-y_0) \Leftrightarrow F(u,v)e^{-j2\pi(ux_0/M+vy_0/N)}$<br>When $x_0 = u_0 = M/2$ and $y_0 = v_0 = N/2$, then<br>$f(x,y)(-1)^{x+y} \Leftrightarrow F(u-M/2, v-N/2)$<br>$f(x-M/2, y-N/2) \Leftrightarrow F(u,v)(-1)^{u+v}$ |

| | |
|---|---|
| Conjugate symmetry | $F(u,v) = F^*(-u,-v)$ <br> $\lvert F(u,v)\rvert = \lvert F(-u,-v)\rvert$ |
| Differentiation | $\dfrac{\partial^n f(x,y)}{\partial x^n} \Leftrightarrow (ju)^n F(u,v)$ <br> $(-jx)^n f(x,y) \Leftrightarrow \dfrac{\partial^n F(u,v)}{\partial u^n}$ |
| Laplacian | $\nabla^2 f(x,y) \Leftrightarrow -(u^2 + v^2)F(u,v)$ |
| Distributivity | $\Im[f_1(x,y) + f_2(x,y)] = \Im[f_1(x,y)] + \Im[f_2(x,y)]$ <br> $\Im[f_1(x,y) \cdot f_2(x,y)] \neq \Im[f_1(x,y)] \cdot \Im[f_2(x,y)]$ |
| Scaling | $af(x,y) \Leftrightarrow aF(u,v), f(ax,by) \Leftrightarrow \dfrac{1}{\lvert ab\rvert} F(u/a, v/b)$ |
| Rotation | $x = r\cos\theta \quad y = r\sin\theta \quad u = \omega\cos\varphi \quad v = \omega\sin\varphi$ <br> $f(r, \theta + \theta_0) \Leftrightarrow F(\omega, \varphi + \theta_0)$ |
| Periodicity | $F(u,v) = F(u+M,v) = F(u,v+N) = F(u+M,v+N)$ <br> $f(x,y) = f(x+M,y) = f(x,y+N) = f(x+M,y+N)$ |
| Separability | See Eqs. (4.6-14) and (4.6-15). Separability implies that we can compute the 2-D transform of an image by first computing 1-D transforms along each row of the image, and then computing a 1-D transform along each column of this intermediate result. The reverse, columns and then rows, yields the same result. |

# 2D Aliasing

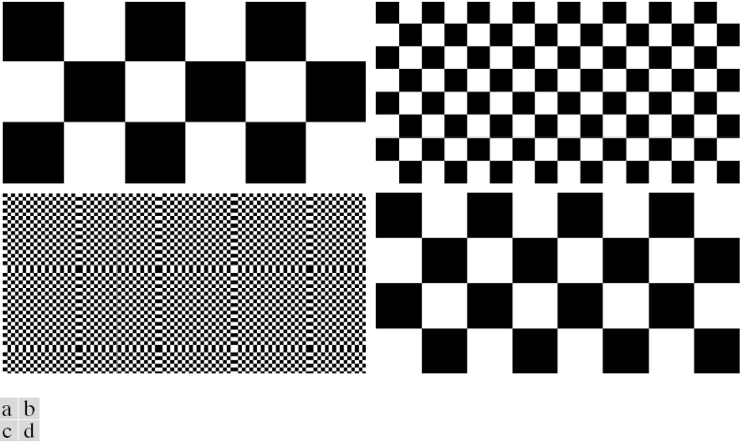
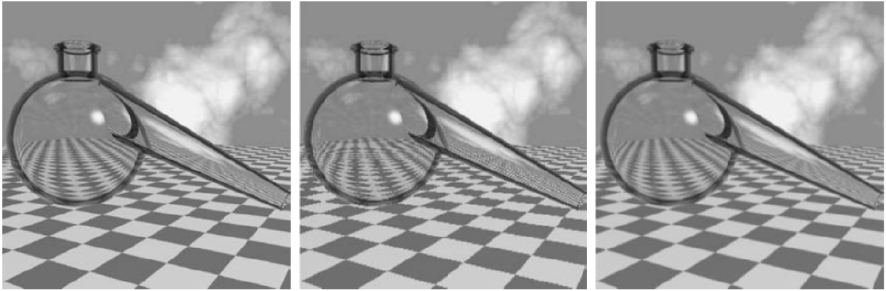



**FIGURE 4.16** Aliasing in images. In (a) and (b), the lengths of the sides of the squares are 16 and 6 pixels, respectively, and aliasing is visually negligible. In (c) and (d), the sides of the squares are 0.9174 and 0.4798 pixels, respectively, and the results show significant aliasing. Note that (d) masquerades as a "normal" image.

a b c

**FIGURE 4.17** Illustration of aliasing on resampled images. (a) A digital image with negligible visual aliasing. (b) Result of resizing the image to 50% of its original size by pixel deletion. Aliasing is clearly visible. (c) Result of blurring the image in (a) with a $3 \times 3$ averaging filter prior to resizing. The image is slightly more blurred than (b), but aliasing is not longer objectionable. (Original image courtesy of the Signal Compression Laboratory, University of California, Santa Barbara.)

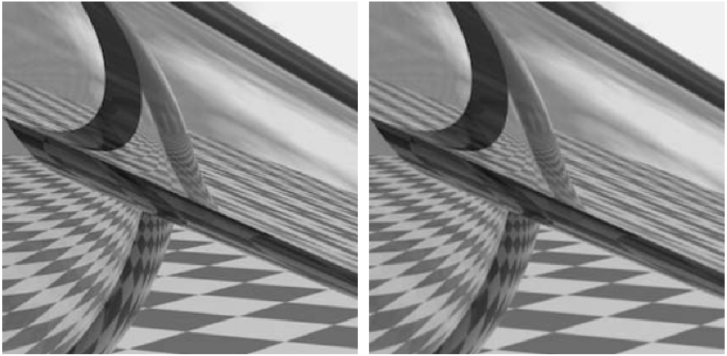
# 2D Aliasing



a b c

**FIGURE 4.18** Illustration of jaggies. (a) A 1024 × 1024 digital image of a computer-generated scene with negligible visible aliasing. (b) Result of reducing (a) to 25% of its original size using bilinear interpolation. (c) Result of blurring the image in (a) with a 5 × 5 averaging filter prior to resizing it to 25% using bilinear interpolation. (Original image courtesy of D. P. Mitchell, Mental Landscape, LLC.)

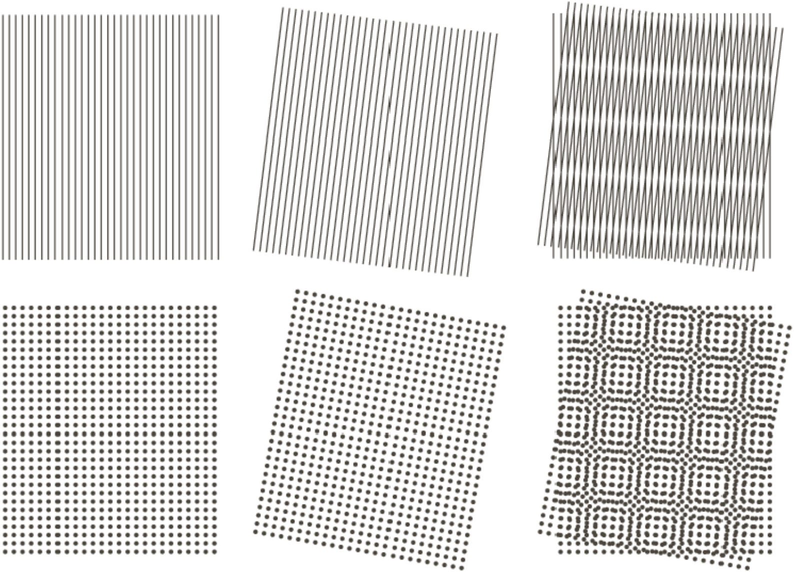
# 2D Aliasing



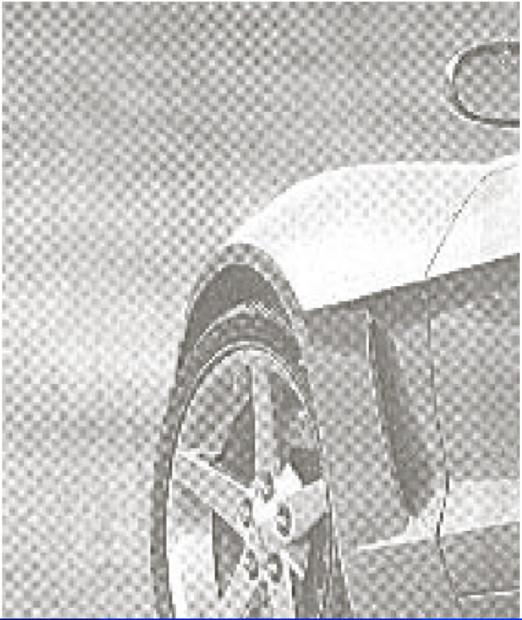
a b

**FIGURE 4.19** Image zooming. (a) A 1024 × 1024 digital image generated by pixel replication from a 256 × 256 image extracted from the middle of Fig. 4.18(a). (b) Image generated using bi-linear interpolation, showing a significant reduction in jaggies.

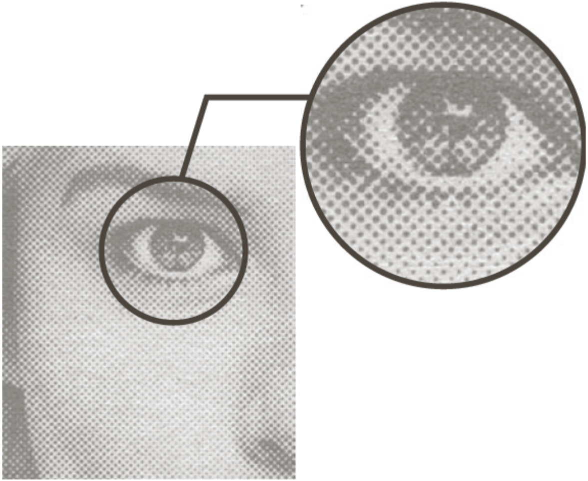
# Moiré



# Moiré



# Moiré

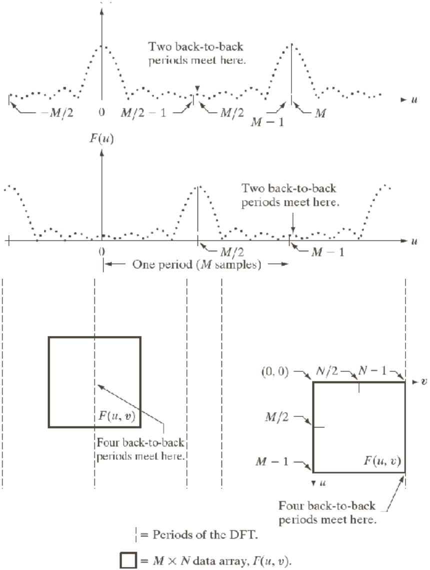


# TFD-2D Properties

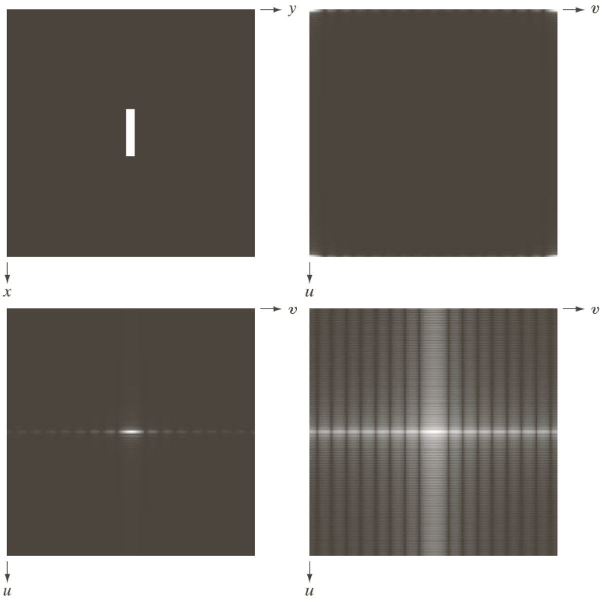
| | Spatial Domain† | | Frequency Domain† |
|---|---|---|---|
| 1) | $f(x, y)$ real | $\Leftrightarrow$ | $F^*(u, v) = F(-u, -v)$ |
| 2) | $f(x, y)$ imaginary | $\Leftrightarrow$ | $F^*(-u, -v) = -F(u, v)$ |
| 3) | $f(x, y)$ real | $\Leftrightarrow$ | $R(u, v)$ even; $I(u, v)$ odd |
| 4) | $f(x, y)$ imaginary | $\Leftrightarrow$ | $R(u, v)$ odd; $I(u, v)$ even |
| 5) | $f(-x, -y)$ real | $\Leftrightarrow$ | $F^*(u, v)$ complex |
| 6) | $f(-x, -y)$ complex | $\Leftrightarrow$ | $F(-u, -v)$ complex |
| 7) | $f^*(x, y)$ complex | $\Leftrightarrow$ | $F^*(-u - v)$ complex |
| 8) | $f(x, y)$ real and even | $\Leftrightarrow$ | $F(u, v)$ real and even |
| 9) | $f(x, y)$ real and odd | $\Leftrightarrow$ | $F(u, v)$ imaginary and odd |
| 10) | $f(x, y)$ imaginary and even | $\Leftrightarrow$ | $F(u, v)$ imaginary and even |
| 11) | $f(x, y)$ imaginary and odd | $\Leftrightarrow$ | $F(u, v)$ real and odd |
| 12) | $f(x, y)$ complex and even | $\Leftrightarrow$ | $F(u, v)$ complex and even |
| 13) | $f(x, y)$ complex and odd | $\Leftrightarrow$ | $F(u, v)$ complex and odd |

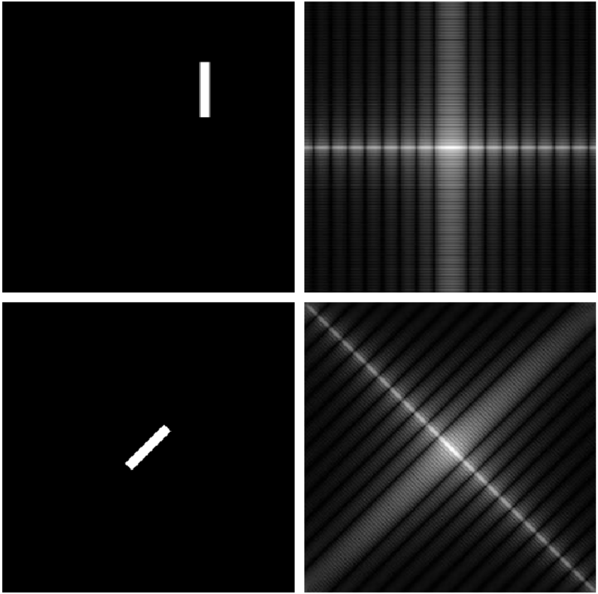
†Recall that $x, y, u$, and $v$ are *discrete* (integer) variables, with $x$ and $u$ in the range $[0, M - 1]$, and $y$, and $v$ in the range $[0, N - 1]$. To say that a complex function is *even* means that its real *and* imaginary parts are even, and similarly for an odd complex function.

# TFD-2D Properties

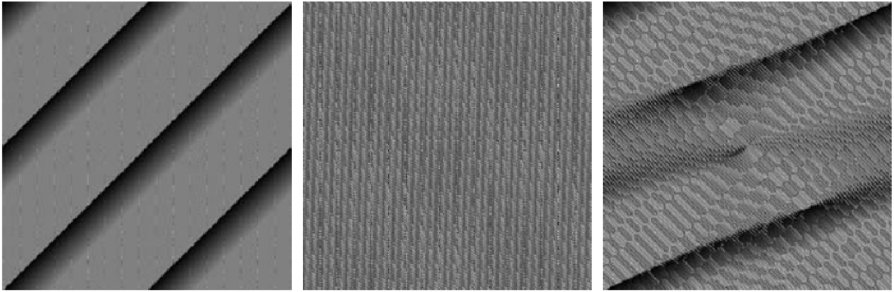


# TFD-2D Properties





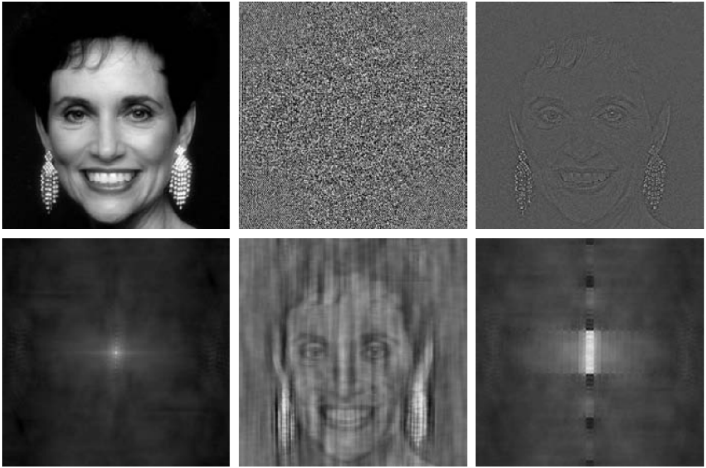
# TFD-2D Properties



a b c

**FIGURE 4.26** Phase angle array corresponding (a) to the image of the centered rectangle in Fig. 4.24(a), (b) to the translated image in Fig. 4.25(a), and (c) to the rotated image in Fig. 4.25(c).

# TFD-2D Properties



a b c
d e f

**FIGURE 4.27** (a) Woman. (b) Phase angle. (c) Woman reconstructed using only the phase angle. (d) Woman reconstructed using only the spectrum. (e) Reconstruction using the phase angle corresponding to the woman and the spectrum corresponding to the rectangle in Fig. 4.24(a). (f) Reconstruction using the phase of the rectangle and the

# TFD-2D Properties

| Name | DFT Pairs |
| --- | --- |
| 1) Symmetry properties | See Table 4.1 |
| 2) Linearity | $af_1(x,y) + bf_2(x,y) \Leftrightarrow aF_1(u,v) + bF_2(u,v)$ |
| 3) Translation (general) | $f(x,y)e^{j2\pi(u_0x/M+v_0y/N)} \Leftrightarrow F(u-u_0, v-v_0)$<br>$f(x-x_0, y-y_0) \Leftrightarrow F(u,v)e^{-j2\pi(ux_0/M+vy_0/N)}$ |
| 4) Translation to center of the frequency rectangle, $(M/2, N/2)$ | $f(x,y)(-1)^{x+y} \Leftrightarrow F(u-M/2, v-N/2)$<br>$f(x-M/2, y-N/2) \Leftrightarrow F(u,v)(-1)^{u+v}$ |
| 5) Rotation | $f(r, \theta+\theta_0) \Leftrightarrow F(\omega, \varphi+\theta_0)$<br>$x = r\cos\theta \quad y = r\sin\theta \quad u = \omega\cos\varphi \quad v = \omega\sin\varphi$ |
| 6) Convolution theorem† | $f(x,y) \star h(x,y) \Leftrightarrow F(u,v)H(u,v)$<br>$f(x,y)h(x,y) \Leftrightarrow F(u,v) \star H(u,v)$ |

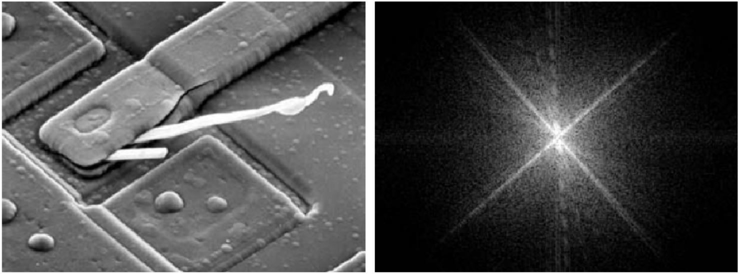
(*Continued*)

# TFD-2D Properties

| Name | DFT Pairs |
| --- | --- |
| 7) Correlation theorem† | $f(x,y) \star h(x,y) \Leftrightarrow F^*(u,v)\,H(u,v)$<br>$f^*(x,y)h(x,y) \Leftrightarrow F(u,v) \star H(u,v)$ |
| 8) Discrete unit impulse | $\delta(x,y) \Leftrightarrow 1$ |
| 9) Rectangle | $\text{rect}[a,b] \Leftrightarrow ab\dfrac{\sin(\pi ua)}{(\pi ua)}\dfrac{\sin(\pi vb)}{(\pi vb)}e^{-j\pi(ua+vb)}$ |
| 10) Sine | $\sin(2\pi u_0 x + 2\pi v_0 y) \Leftrightarrow$<br>$j\dfrac{1}{2}\left[\delta(u + Mu_0, v + Nv_0) - \delta(u - Mu_0, v - Nv_0)\right]$ |
| 11) Cosine | $\cos(2\pi u_0 x + 2\pi v_0 y) \Leftrightarrow$<br>$\dfrac{1}{2}\left[\delta(u + Mu_0, v + Nv_0) + \delta(u - Mu_0, v - Nv_0)\right]$ |

The following Fourier transform pairs are derivable only for continuous variables, denoted as before by $t$ and $z$ for spatial variables and by $\mu$ and $\nu$ for frequency variables. These results can be used for DFT work by sampling the continuous forms.

| | |
| --- | --- |
| 12) *Differentiation* (The expressions on the right assume that $f(\pm\infty, \pm\infty) = 0$.) | $\left(\dfrac{\partial}{\partial t}\right)^m\left(\dfrac{\partial}{\partial z}\right)^n f(t,z) \Leftrightarrow (j2\pi\mu)^m(j2\pi\nu)^n F(\mu,\nu)$<br>$\dfrac{\partial^m f(t,z)}{\partial t^m} \Leftrightarrow (j2\pi\mu)^m F(\mu,\nu); \dfrac{\partial^n f(t,z)}{\partial z^n} \Leftrightarrow (j2\pi\nu)^n F(\mu,\nu)$ |
| 13) *Gaussian* | $A2\pi\sigma^2 e^{-2\pi^2\sigma^2(t^2+z^2)} \Leftrightarrow Ae^{-(\mu^2+\nu^2)/2\sigma^2}$ ($A$ is a constant) |



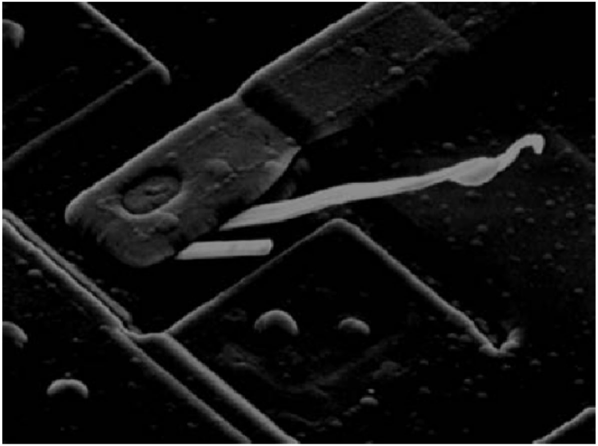
a b

**FIGURE 4.29** (a) SEM image of a damaged integrated circuit. (b) Fourier spectrum of (a). (Original image courtesy of Dr. J. M. Hudak, Brockhouse Institute for Materials Research, McMaster University, Hamilton, Ontario, Canada.)

# Filters

$$G(u, v) = H(u, v) \cdot F(u, v)$$

Filter result, after deleting $F(M/2, N/2)$ – eliminating the DC level.



Low-Pass High-Pass High-Pass + DC

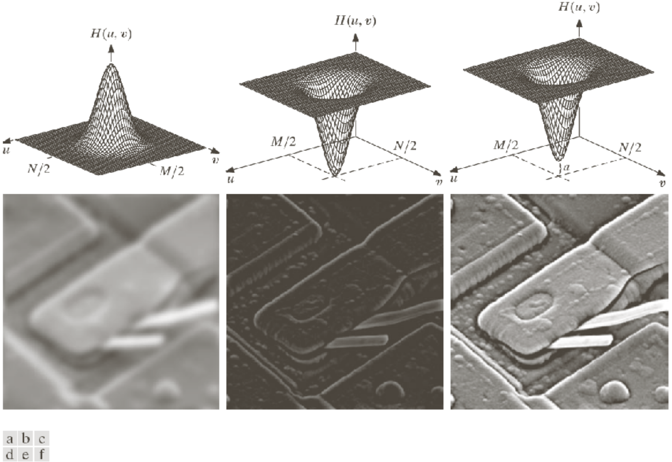



**FIGURE 4.31** Top row: frequency domain filters. Bottom row: corresponding filtered images obtained using Eq. (4.7-1). We used $a = 0.85$ in (c) to obtain (f) (the height of the filter itself is 1). Compare (f) with Fig. 4.29(a).

# Padding

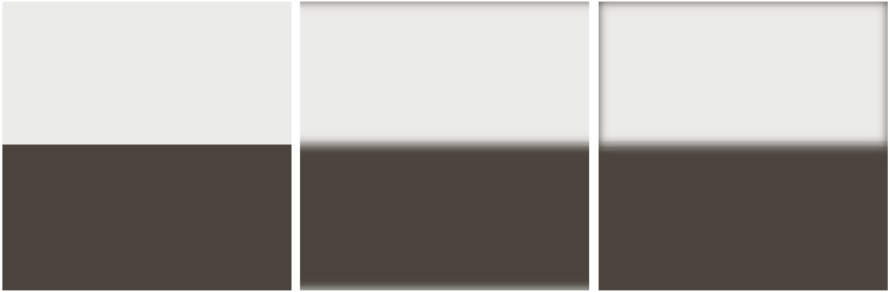
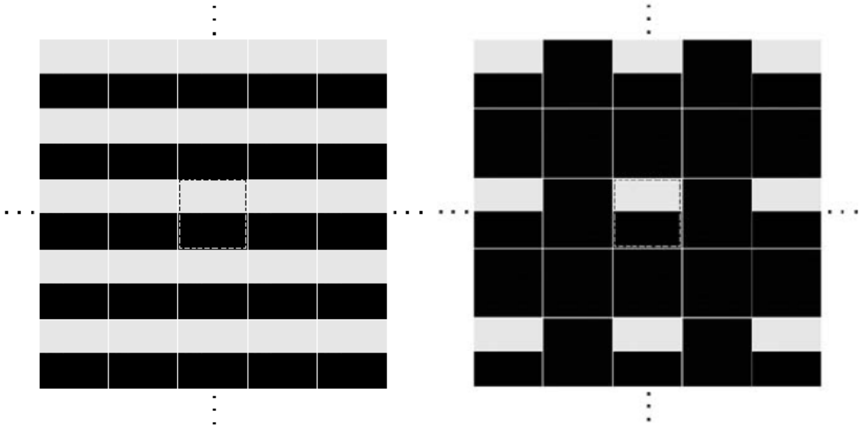


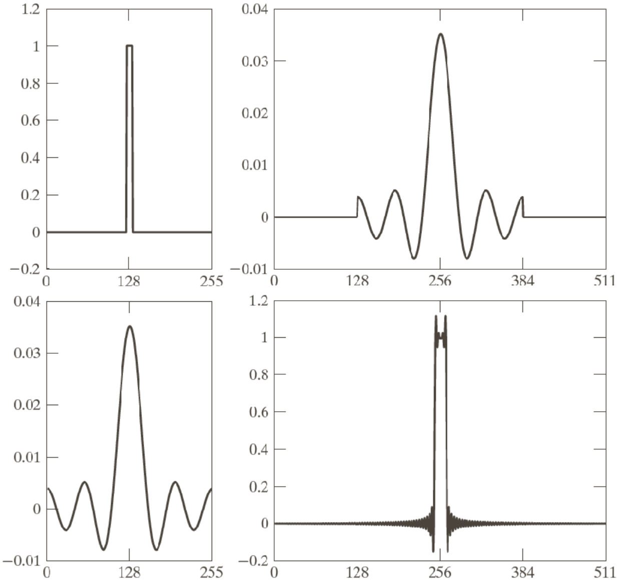

**FIGURE 4.32** (a) A simple image. (b) Result of blurring with a Gaussian lowpass filter without padding. (c) Result of lowpass filtering with padding. Compare the light area of the vertical edges in (b) and (c).



a b

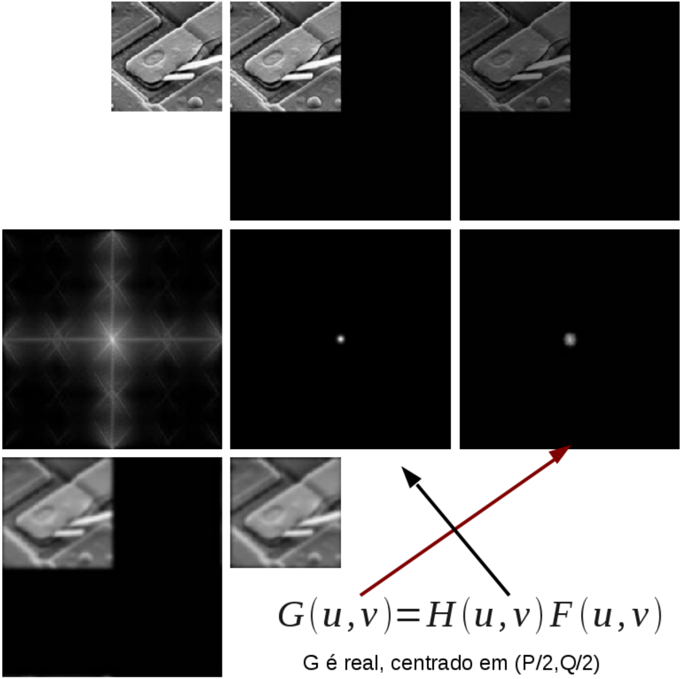
**FIGURE 4.33** 2-D image periodicity inherent in using the DFT. (a) Periodicity without image padding. (b) Periodicity after padding with 0s (black). The dashed areas in the center correspond to the image in Fig. 4.32(a). (The thin white lines in both images are superimposed for clarity; they are not part of the data.)

# Periodicity



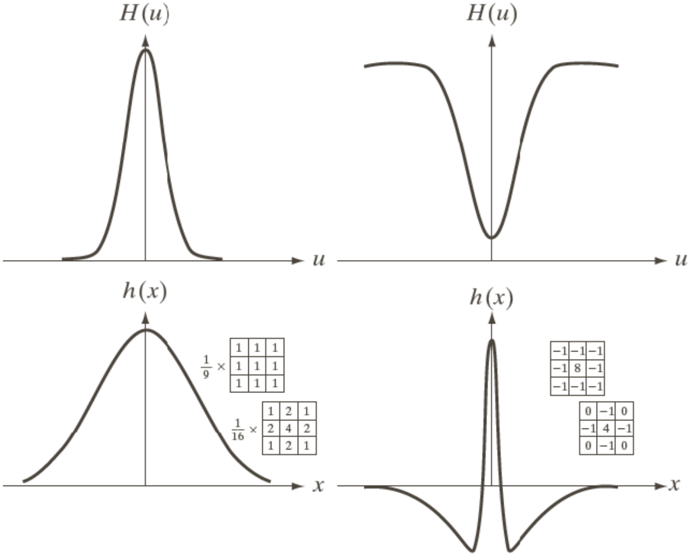

a c
b d

**FIGURE 4.34** (a) Original filter specified in the (centered) frequency domain. (b) Spatial representation obtained by computing the IDFT of (a). (c) Result of padding (b) to twice its length (note the discontinuities). (d) Corresponding filter in the frequency domain obtained by computing the DFT of (c). Note the ringing caused by the discontinuities in (c). (The curves appear continuous because the points were joined to simplify visual analysis.)




a b c
d e f
g h

**FIGURE 4.36**
(a) An $M \times N$ image, $f$. (b) Padded image, $f_p$ of size $P \times Q$. (c) Result of multiplying $f_p$ by $(-1)^{x+y}$. (d) Spectrum of $F_p$. (e) Centered Gaussian lowpass filter, $H$, of size $P \times Q$. (f) Spectrum of the product $HF_p$. (g) $g_p$, the product of $(-1)^{x+y}$ and the real part of the IDFT of $HF_p$. (h) Final result, $g$, obtained by cropping the first $M$ rows and $N$ columns of $g_p$.



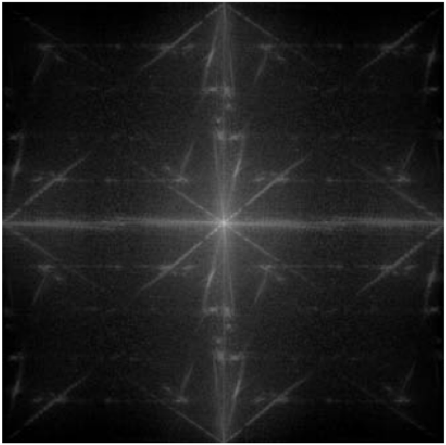

a c
b d

**FIGURE 4.37**
(a) A 1-D Gaussian lowpass filter in the frequency domain. (b) Spatial lowpass filter corresponding to (a). (c) Gaussian highpass filter in the frequency domain. (d) Spatial highpass filter corresponding to (c). The small 2-D masks shown are spatial filters we used in Chapter 3.

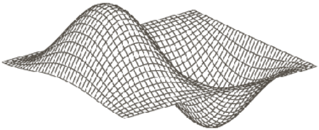
# Example

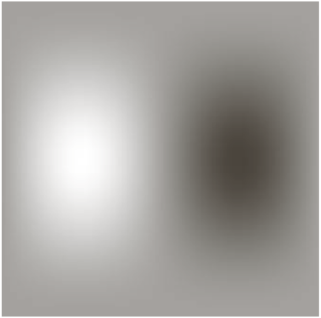
a b

**FIGURE 4.38**
(a) Image of a building, and (b) its spectrum.



| | | |
|---|---|---|
| −1 | 0 | 1 |
| −2 | 0 | 2 |
| −1 | 0 | 1 |





a b
c d

**FIGURE 4.39** (a) A spatial mask and perspective plot of its corresponding frequency domain filter. (b) Filter shown as an image. (c) Result of filtering Fig. 4.38(a) in the frequency domain with the filter in (b). (d) Result of filtering the same image with the spatial filter in (a). The results are identical.

# Low-Pass Filters

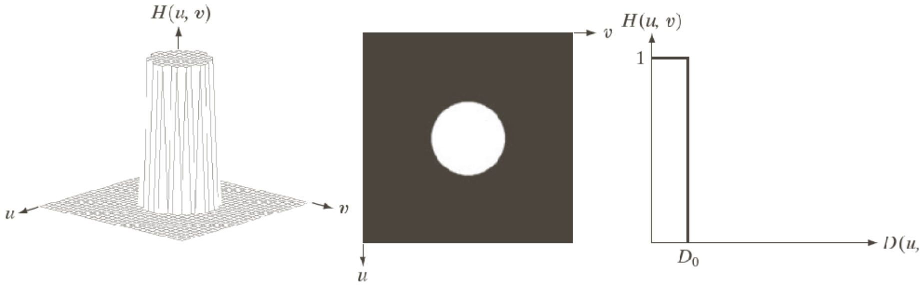
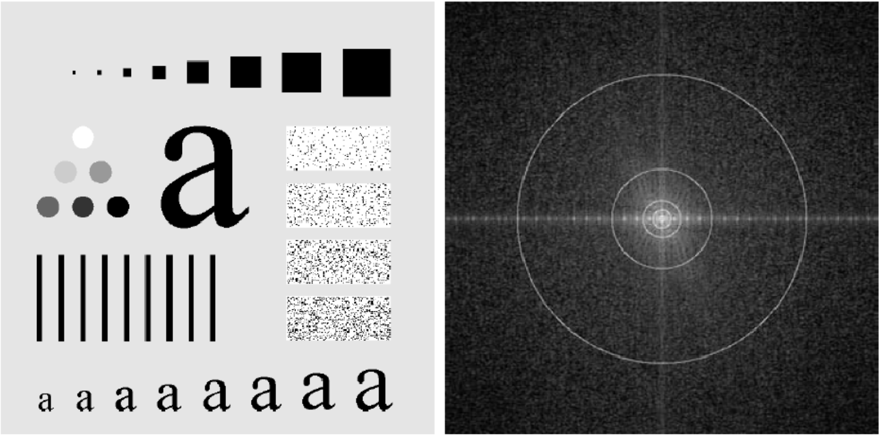



**FIGURE 4.40** (a) Perspective plot of an ideal lowpass-filter transfer function. (b) Filter displayed as an im
(c) Filter radial cross section.




a b

**FIGURE 4.41** (a) Test pattern of size 688 × 688 pixels, and (b) its Fourier spectrum. The spectrum is double the image size due to padding but is shown in half size so that it fits in the page. The superimposed circles have radii equal to 10, 30, 60, 160, and 460 with respect to the full-size spectrum image. These radii enclose 87.0, 93.1, 95.7, 97.8, and 99.2% of the padded image power, respectively.

# Ideal Filter

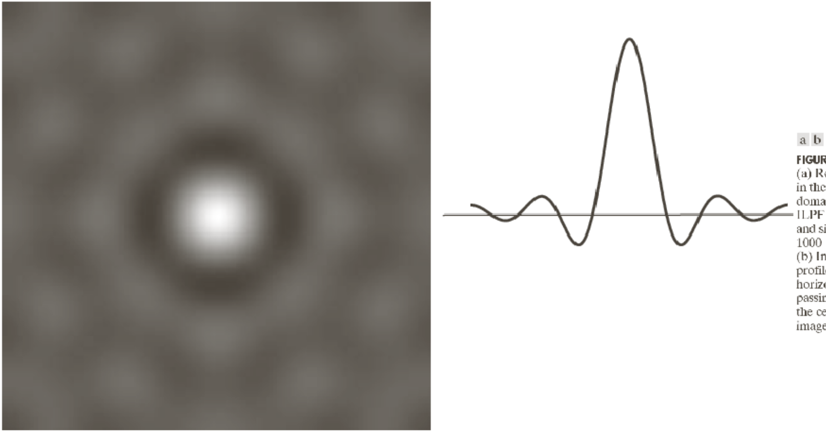
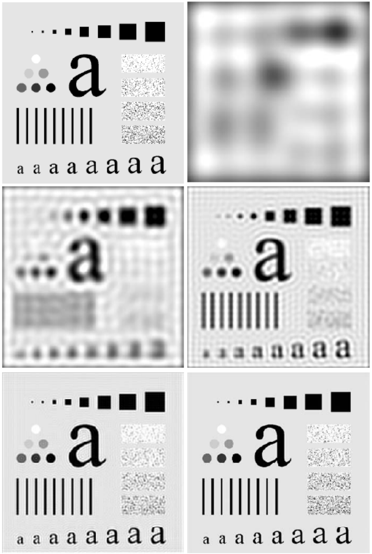



**FIGURE 4.43** (a) Representation in the spatial domain of an ILPF of radius 5 and size $1000 \times 1000$. (b) Intensity profile of a horizontal line passing through the center of the image.

# Ideal Filter



**Filter Ideal:** Cutoff frequencies: radius 10, 30, 60, 160 e 460. Power energy removed by these filter: 13, 6.9, 4.3, 2.2, e 0.8% of the total energy.

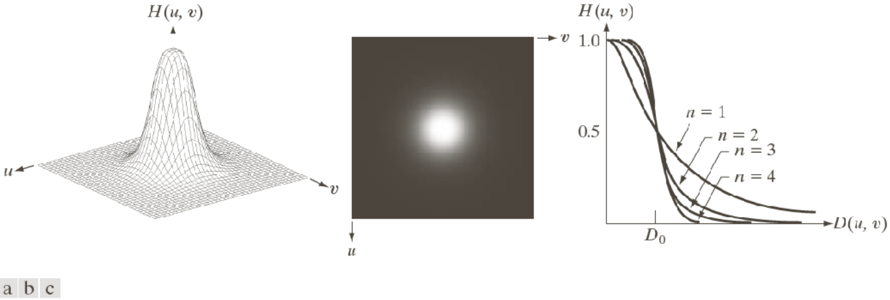



**FIGURE 4.44** (a) Perspective plot of a Butterworth lowpass-filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections of orders 1 through 4.

$$H(u,v) = \frac{1}{1 + [D(u,v)/D_o]^{2n}}$$

In time:

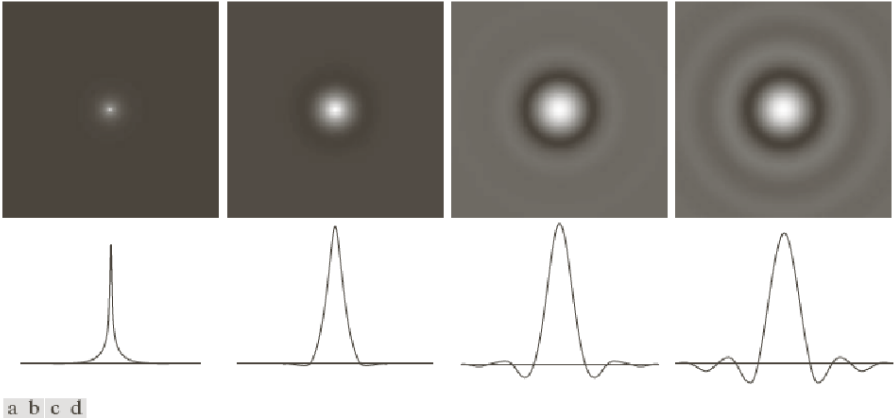
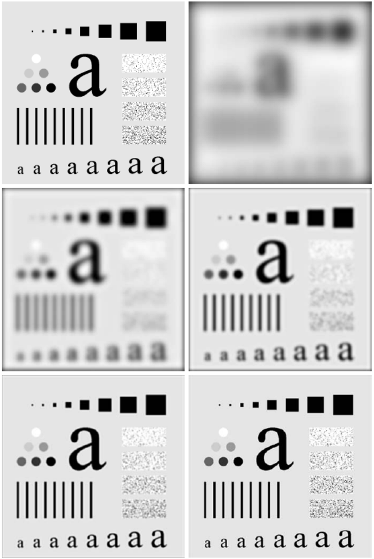


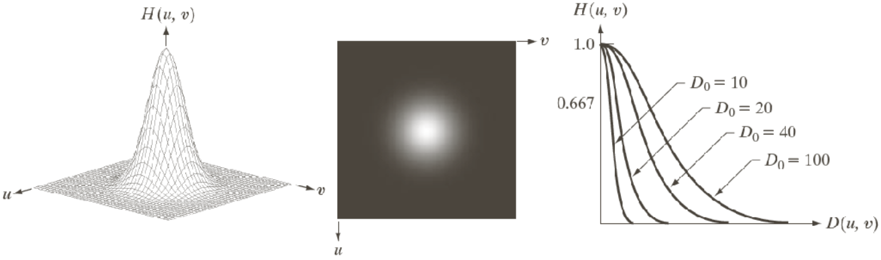

**FIGURE 4.46** (a)–(d) Spatial representation of BLPFs of order 1, 2, 5, and 20, and corresponding intensity profiles through the center of the filters (the size in all cases is 1000 × 1000 and the cutoff frequency is 5). Observe how ringing increases as a function of filter order.

# Butterworth Filter



**Butterworth Filter:** Cutoff frequencies with radius equal to 10, 30, 60, 160 and 460. Power energy removed by these Filters: 13, 6.9, 4.3, 2.2, e 0.8% of the total energy.

# Gaussian Filters

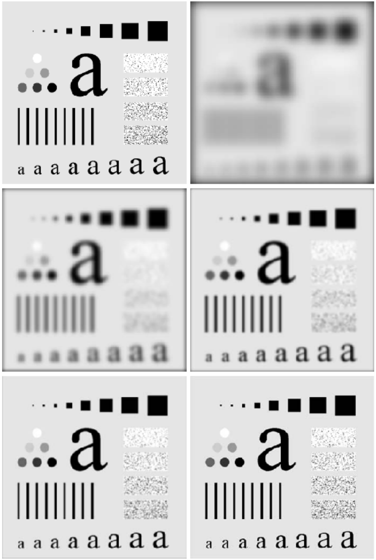



a b c

**FIGURE 4.47** (a) Perspective plot of a GLPF transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections for various values of $D_0$.

$$H(u,v) = e^{-D^2(u,v)/2D_o^2}$$

# Gaussian Filters

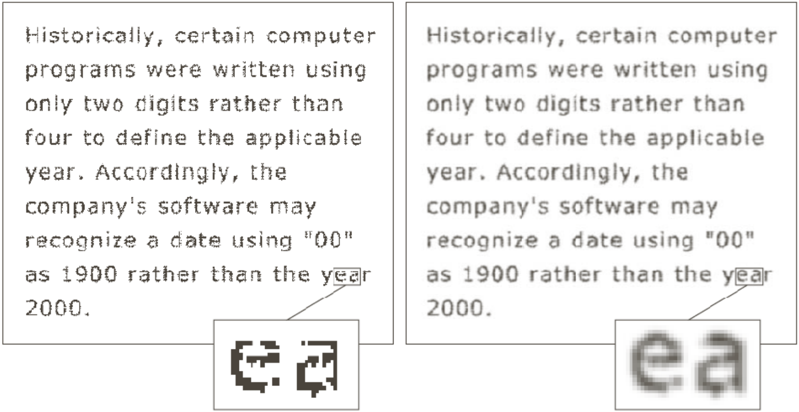


**Gaussian Filter:** Cutoff frequencies with radius equal to 10, 30, 60, 160 and 460. Power energy removed by these Filters: 13, 6.9, 4.3, 2.2, e 0.8% of the total energy.

# Summary – Filters

**TABLE 4.4**
Lowpass filters. $D_0$ is the cutoff frequency and $n$ is the order of the Butterworth filter.

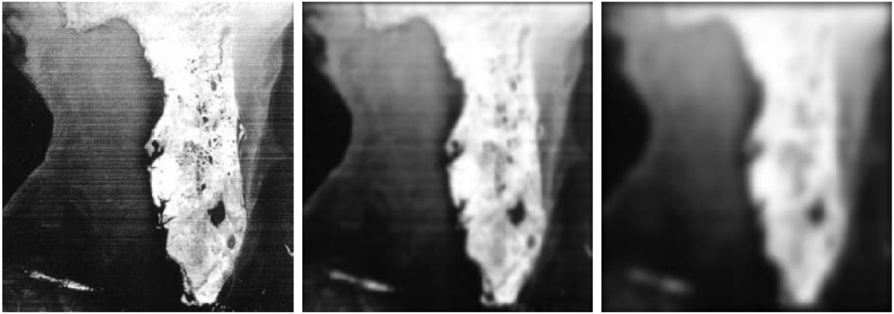
| Ideal | Butterworth | Gaussian |
|---|---|---|
| $H(u,v) = \begin{cases} 1 & \text{if } D(u,v) \leq D_0 \\ 0 & \text{if } D(u,v) > D_0 \end{cases}$ | $H(u,v) = \dfrac{1}{1 + [D(u,v)/D_0]^{2n}}$ | $H(u,v) = e^{-D^2(u,v)/2D_0^2}$ |




a b

**FIGURE 4.49** (a) Sample text of low resolution (note broken characters in magnified view). (b) Result of filtering with a GLPF (broken character segments were joined).


**FIGURE 4.50** (a) Original image (784 × 732 pixels). (b) Result of filtering using a GLPF with $D_0 = 100$. (c) Result of filtering using a GLPF with $D_0 = 80$. Note the reduction in fine skin lines in the magnified sections in (b) and (c).



a b c

**FIGURE 4.51** (a) Image showing prominent horizontal scan lines. (b) Result of filtering using a GLPF with $D_0 = 50$. (c) Result of using a GLPF with $D_0 = 20$. (Original image courtesy of NOAA.)

**High-Pass Filters:**

$$H_{HP}(u,v) = 1 - H_{LP}(u,v)$$

**TABLE 4.5**
Highpass filters. $D_0$ is the cutoff frequency and $n$ is the order of the Butterworth filter.

| Ideal | Butterworth | Gaussian |
|---|---|---|
| $H(u,v) = \begin{cases} 1 & \text{if } D(u,v) \leq D_0 \\ 0 & \text{if } D(u,v) > D_0 \end{cases}$ | $H(u,v) = \dfrac{1}{1 + [D_0/D(u,v)]^{2n}}$ | $H(u,v) = 1 - e^{-D^2(u,v)/2D_0^2}$ |

# High-Pass Filters

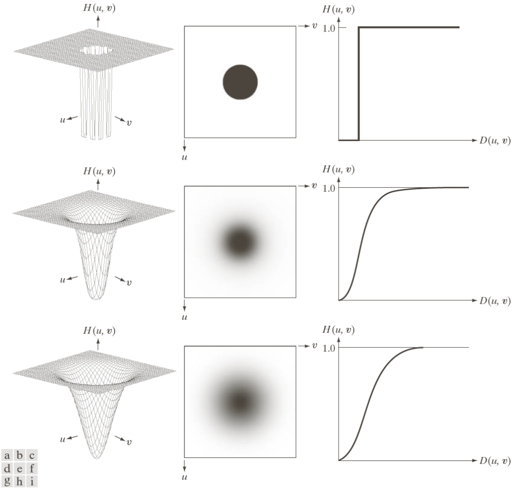



**FIGURE 4.52** Top row: Perspective plot, image representation, and cross section of a typical ideal highpass filter. Middle and bottom rows: The same sequence for typical Butterworth and Gaussian highpass filters.

Ideal

Butterworth

Gaussian

# Spatial Representation

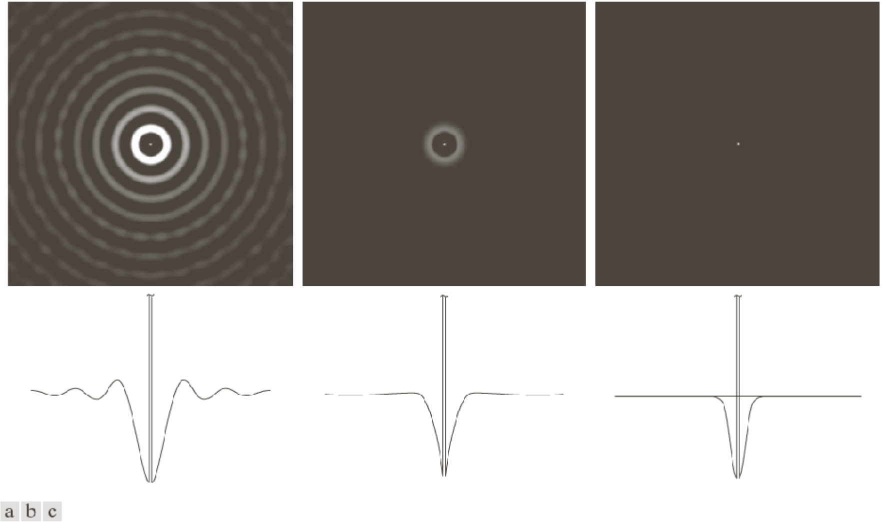



**FIGURE 4.53** Spatial representation of typical (a) ideal, (b) Butterworth, and (c) Gaussian frequency domain highpass filters, and corresponding intensity profiles through their centers.

# Ideal High-Pass Filters

a b c

**FIGURE 4.54** Results of highpass filtering the image in Fig. 4.41(a) using an IHPF with $D_0 = 30, 60,$ and 160.

# Butterworth High-Pass Filters

a b c

**FIGURE 4.55** Results of highpass filtering the image in Fig. 4.41(a) using a BHPF of order 2 with $D_0 = 30, 60,$ and 160, corresponding to the circles in Fig. 4.41(b). These results are much smoother than those obtained with an IHPF.

# Gaussian High-Pass Filters

a b c

**FIGURE 4.56** Results of highpass filtering the image in Fig. 4.41(a) using a GHPF with $D_0 = 30, 60$, and 160, corresponding to the circles in Fig. 4.41(b). Compare with Figs. 4.54 and 4.55.

# Example

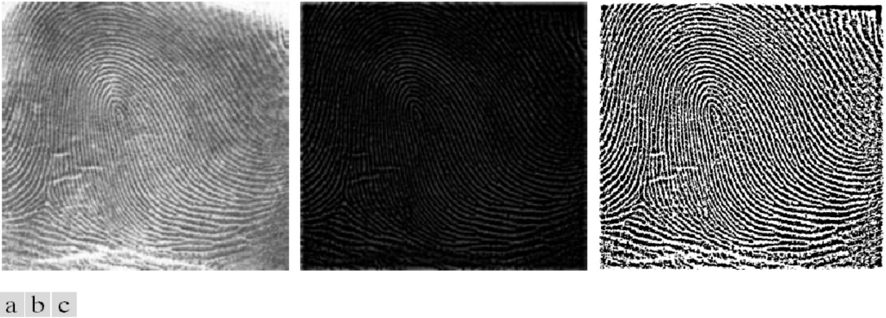



**FIGURE 4.57** (a) Thumb print. (b) Result of highpass filtering (a). (c) Result of thresholding (b). (Original image courtesy of the U.S. National Institute of Standards and Technology.)

# Laplacian Filter

$$g(x, y) = f(x, y) - \nabla^2 f(x, y)$$

$$g(x, y) = F^{-1}\left\{\left[1 + (u - M/2)^2 (v - N/2)^2\right] F(u, v)\right\} f(x, y)$$

# Example

a b

**FIGURE 4.58** (a) Original, blurry image. (b) Image enhanced using the Laplacian in the frequency domain. Compare with Fig. 3.38(e).

# Remembering ...

$$\boxed{g(x,y) = f(x,y) - f_{LP}(x,y)} \quad \boxed{f_{HB}(x,y) = Af(x,y) - f_{LP}(x,y)}$$

$$f_{HB}(x,y) = (A-1)f(x,y) + f(x,y) - f_{LP}(x,y)$$

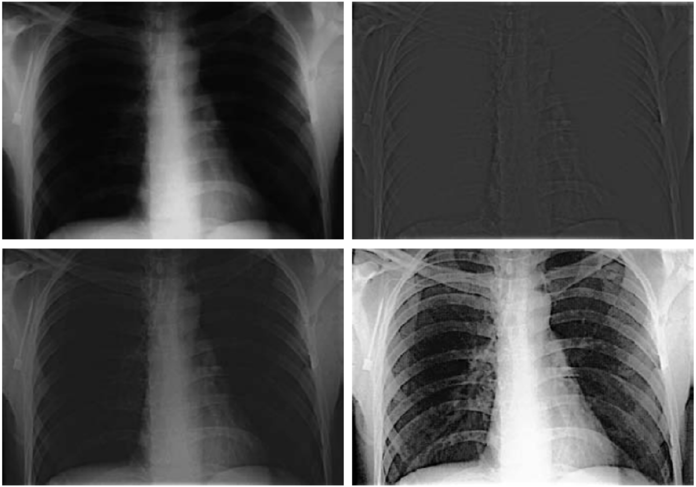
and

$$f_{hb}(x,y) = (A-1)f(x,y) + f_{HP}(x,y)$$

$$\boxed{H_{HP}(u,v) = 1 - H_{LP}(u,v)}$$

$$\boxed{H_{HB}(u,v) = (A-1) + H_{HP}(u,v)}$$

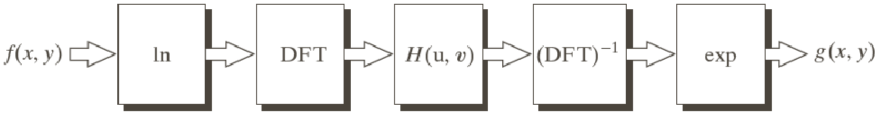
# Example



| a | b |
|---|---|
| c | d |

**FIGURE 4.59** (a) A chest X-ray image. (b) Result of highpass filtering with a Gaussian filter. (c) Result of high-frequency-emphasis filtering using the same filter. (d) Result of performing histogram equalization on (c). (Original image courtesy of Dr. Thomas R.

# Homomorphic Filtering



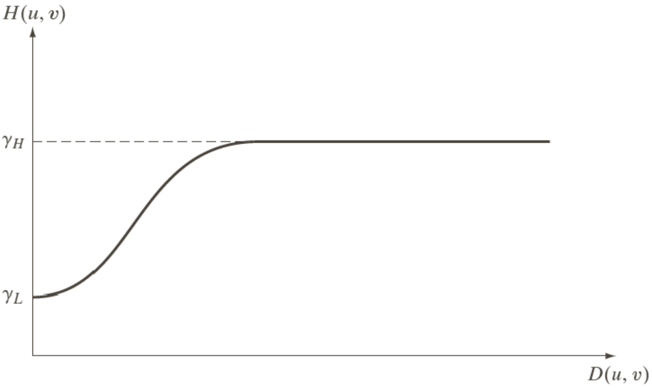




**FIGURE 4.61** Radial cross section of a circularly symmetric homomorphic filter function. The vertical axis is at the center of the frequency rectangle and $D(u, v)$ is the distance from the center.

# Homomorphic Filtering

$$\boxed{f(x,y) = i(x,y) \cdot r(x,y)}$$

But,

$$\text{TF}\{f(x,y)\} \neq \text{TF}\{i(x,y)\} \cdot \text{TF}\{r(x,y)\}$$

# Homomorphic Filtering

$$\boxed{f(x,y) = i(x,y) \cdot r(x,y)}$$

But,

$$\text{TF}\left\{f(x,y)\right\} \neq \text{TF}\left\{i(x,y)\right\} \cdot \text{TF}\left\{r(x,y)\right\}$$

Considering that

$$z(x,y) = \ln f(x,y) = \ln i(x,y) + \ln r(x,y),$$

We have:

$$\text{TF}\left\{z(x,y)\right\} = \text{TF}\left\{\ln f(x,y)\right\} = \text{TF}\left\{\ln i(x,y)\right\} + \text{TF}\left\{\ln r(x,y)\right\}.$$

So:

$$\boxed{Z(u,v) = F_i(u,v) + F_r(u,v)}$$

# Homomorphic Filtering

$$\begin{aligned} S(u,v) &= H(u,v) \cdot Z(u,v) \\ &= H(u,v)F_i(u,v) + H(u,v)F_r(u,v) \end{aligned}$$

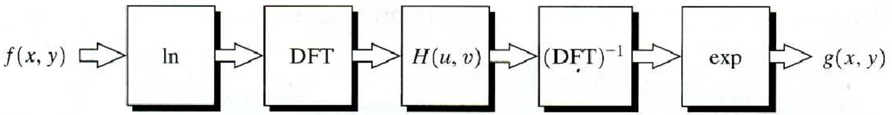
$$\begin{aligned} s(x,y) &= \text{TF}^{-1}\left\{S(u,v)\right\} \\ &= \text{TF}^{-1}\left\{H(u,v)F_i(u,v)\right\} + \text{TF}^{-1}\left\{H(u,v)F_r(u,v)\right\} \end{aligned}$$

$$\begin{aligned} i'(x,y) &= \text{TF}^{-1}\left\{H(u,v)F_i(u,v)\right\} \\ r'(x,y) &= \text{TF}^{-1}\left\{H(u,v)F_r(u,v)\right\} \end{aligned}$$

$$\boxed{s(x,y) = i'(x,y) + r'(x,y)}$$

# Homomorphic Filtering

$$
\begin{aligned}
g(x, y) &= e^{s(x,y)} \\
&= e^{i'(x,y)} \cdot e^{r'(x,y)} \\
&= i_0(x, y) \cdot r_0(x, y)
\end{aligned}
$$

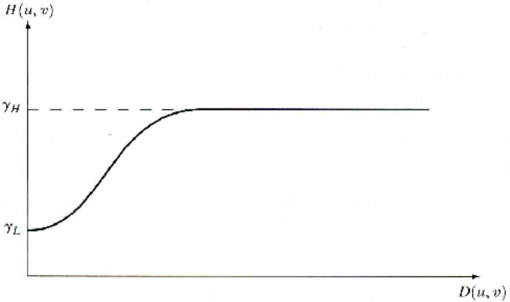


# Homomorphic Filtering

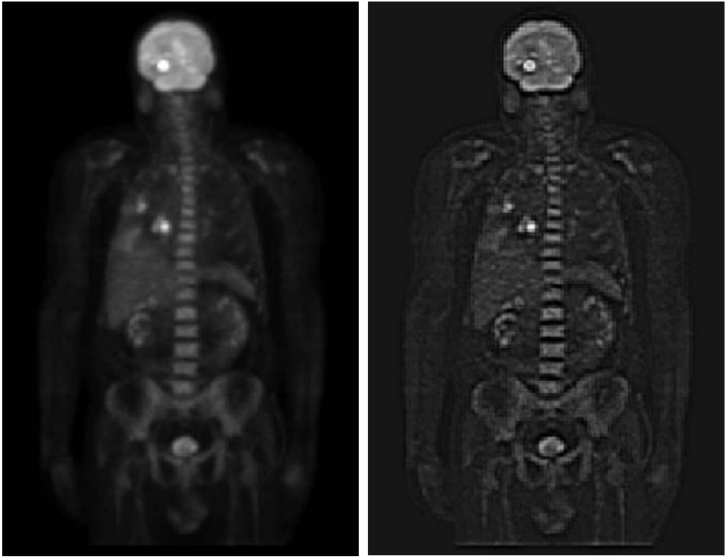
- Illumination - low-frequency spatial variations
- Reflectance  abrupt variations
- Homomorphic Filtering  treat low and high frequencies differently

$$H(u,v) = (\gamma_H - \gamma_L)\left[1 - e^{-c\left(D^2(u,v)/D_0^2\right)}\right] + \gamma_L$$

$$\gamma_L < 1 \text{ and } \gamma_H > 1.$$



# Example: Homomorphic Filtering



a b

**FIGURE 4.62** (a) Full body PET scan. (b) Image enhanced using homomorphic filtering. (Original image courtesy of Dr. Michael E. Casey, CTI PET Systems.)

# Band-reject and Band-Pass Filters

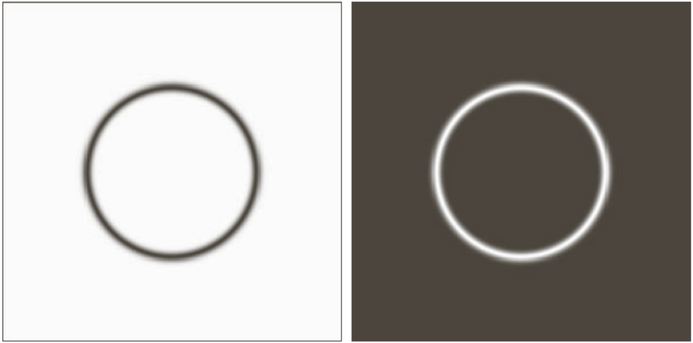
$$H_{BP}(u,v) = 1 - H_{BR}(u,v)$$

**TABLE 4.6**

Bandreject filters. $W$ is the width of the band, $D$ is the distance $D(u, v)$ from the center of the filter, $D_0$ is the cutoff frequency, and $n$ is the order of the Butterworth filter. We show $D$ instead of $D(u, v)$ to simplify the notation in the table.

| Ideal | Butterworth | Gaussian |
|---|---|---|
| $H(u,v) = \begin{cases} 0 & \text{if } D_0 - \frac{W}{2} \le D \le D_0 + \frac{W}{2} \\ 1 & \text{otherwise} \end{cases}$ | $H(u,v) = \dfrac{1}{1 + \left[\dfrac{DW}{D^2 - D_0^2}\right]^{2n}}$ | $H(u,v) = 1 - e^{-\left[\frac{D^2 - D_0^2}{DW}\right]^2}$ |

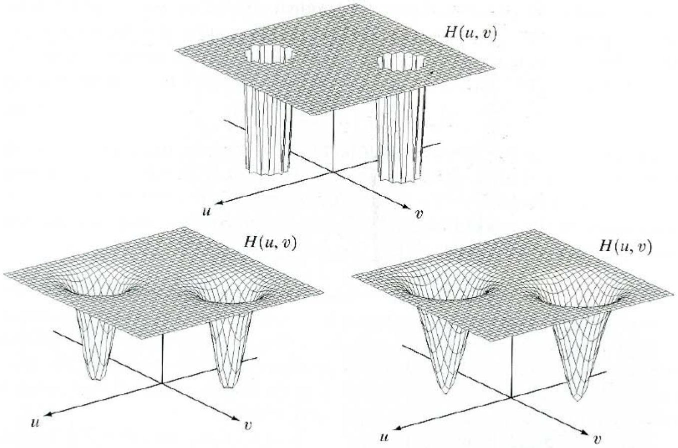
# Band-reject and Band-Pass Filters



a b

**FIGURE 4.63** (a) Bandreject Gaussian filter. (b) Corresponding bandpass filter. The thin black border in (a) was added for clarity; it is not part of the data.

# Notch Filters




a
b c

**FIGURE 5.18** Perspective plots of (a) ideal, (b) Butterworth (of order 2), and (c) Gaussian notch (reject) filters.

# Notch Filters

- Ideal:

$$H(u,v) = \begin{cases} 0, & \text{if} D_1(u,v) \leq D_0 \text{ ou } D_2(u,v) \leq D_0 \\ 1, & \text{otherwise} \end{cases}$$

$$D_1(u,v) = \left[ (u - M/2 - u_0)^2 + (v - N/2 - v_0)^2 \right]^{1/2}$$

$$D_2(u,v) = \left[ (u - M/2 + u_0)^2 + (v - N/2 + v_0)^2 \right]^{1/2}$$

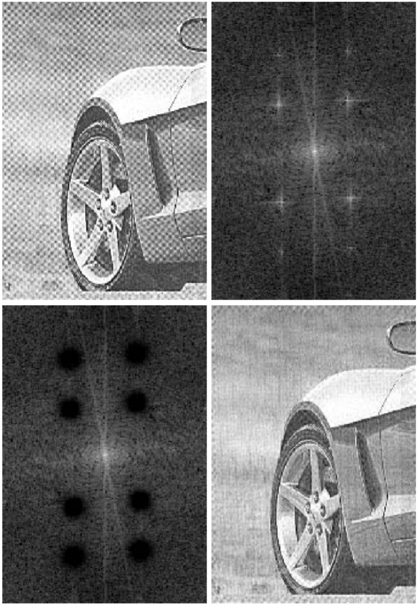
- Butterworth:

$$H(u,v) = \frac{1}{1 + \left[ \frac{D_0^2}{D_1(u,v) D_2(u,v)} \right]^n}$$

- Gaussian:

$$H(u,v) = 1 - e^{-\frac{1}{2} \left[ \frac{D_1(u,v) D_2(u,v)}{D_0^2} \right]}$$

# Notch Filters



| | |
|---|---|
| a | b |
| c | d |

**FIGURE 4.64** (a) Sampled newspaper image showing a moiré pattern. (b) Spectrum. (c) Butterworth notch reject filter multiplied by the Fourier transform. (d) Filtered image.

# Notch Filters

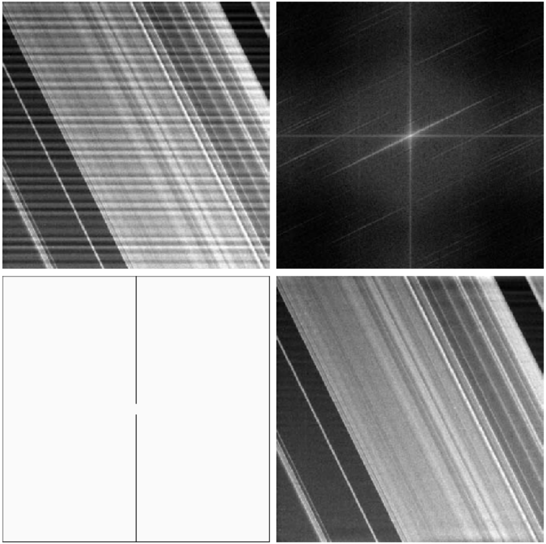
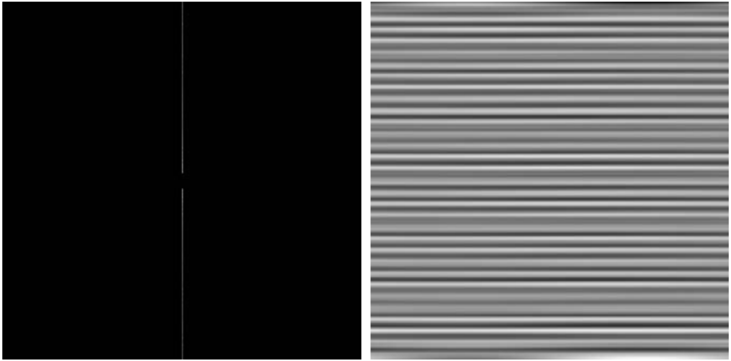



**FIGURE 4.65** (a) 674 × 674 image of the Saturn rings showing nearly periodic interference. (b) Spectrum: The bursts of energy in the vertical axis near the origin correspond to the interference pattern. (c) A vertical notch reject filter. (d) Result of filtering. The thin black border in (c) was added for clarity; it is not part of the data. (Original image courtesy of Dr. Robert A. West, NASA/JPL.)

# Notch Filters



a b

**FIGURE 4.66**
(a) Result (spectrum) of applying a notch pass filter to the DFT of Fig. 4.65(a). (b) Spatial pattern obtained by computing the IDFT of (a).