Image Processing

Transforms

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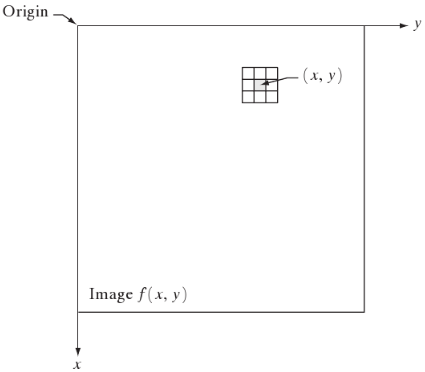
Class 03: Chapter 3 – Spatial Transforms

Spatial Transformation Fucntions

$$g(x, y) = T\left[f(x, y)\right]$$



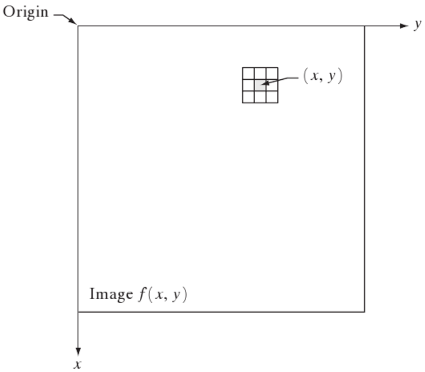
- Linear Operations:

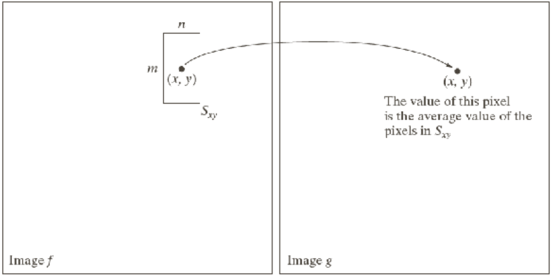
$$H(a{\cdot}f + b{\cdot}g) = a{\cdot}H(f) + b{\cdot}H(g)$$

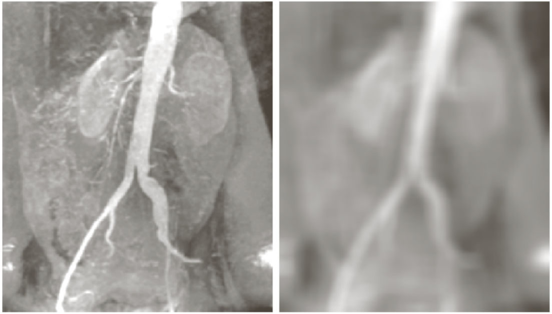
Spatial Transformations

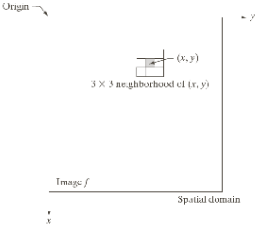
$$g(x, y) = T\left[f(x, y)\right]$$

- Global: all image
- Neighborhood: regions (squares, circles, etc.)
- Pixel-to-pixel: 1×1



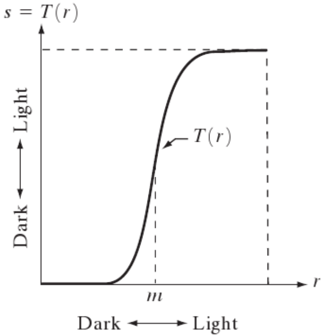

n
m
(x, y)
S_{xy}
(x, y)
The value of this pixel
is the average value of the
pixels in S_{xy}
Image f
Image g

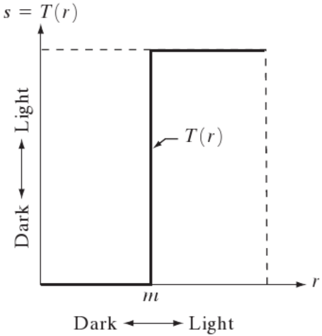



Origin
y
(x, y)
3 × 3 neighborhood of (x, y)
Image f
Spatial domain
x

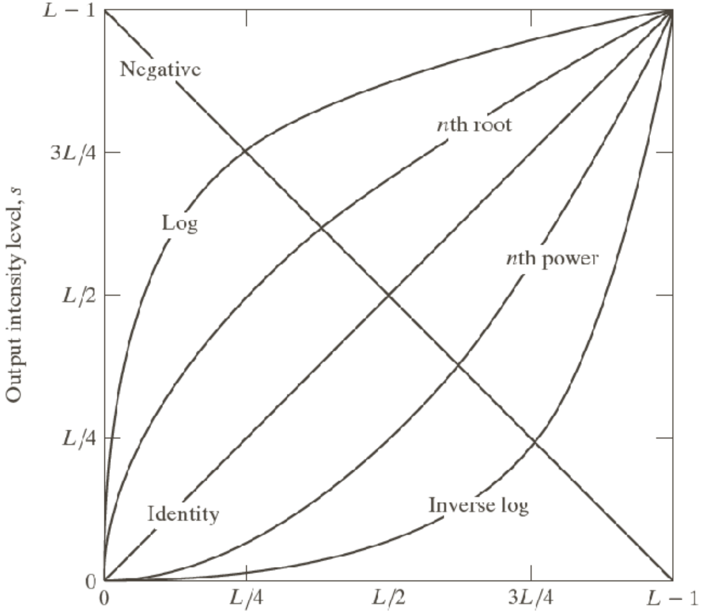
Transformations

$$s = T(r)$$



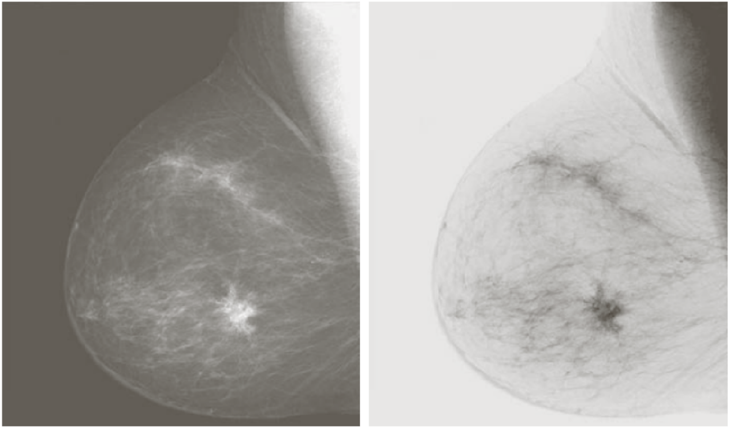


Transformations

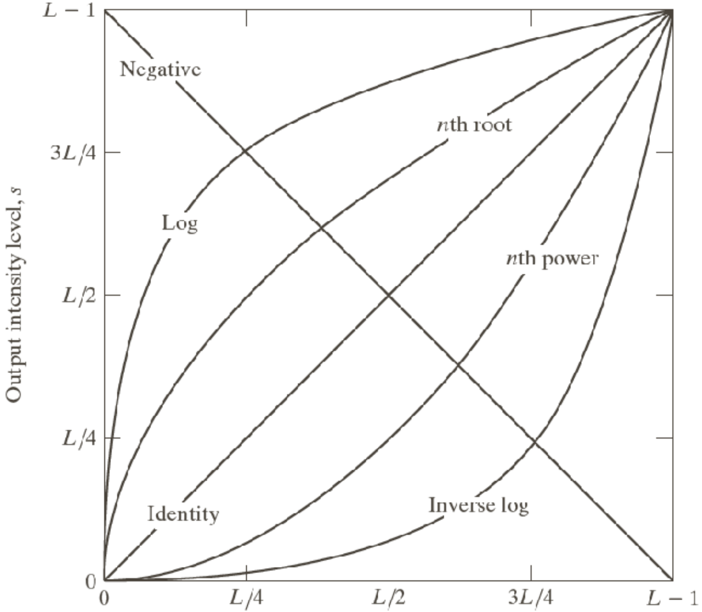


Negative Transformations

$$s = L - 1 - r$$

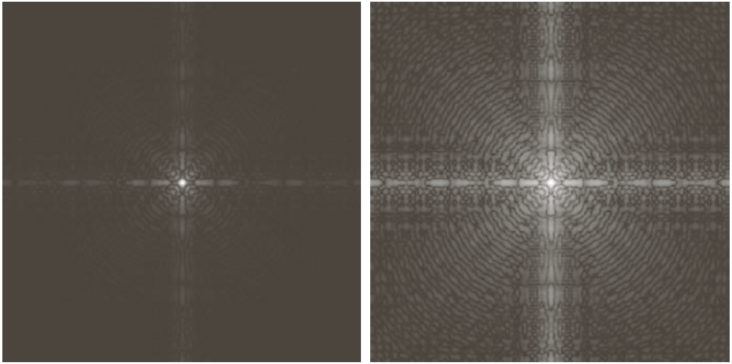


Transformations



Logarithm Transformations

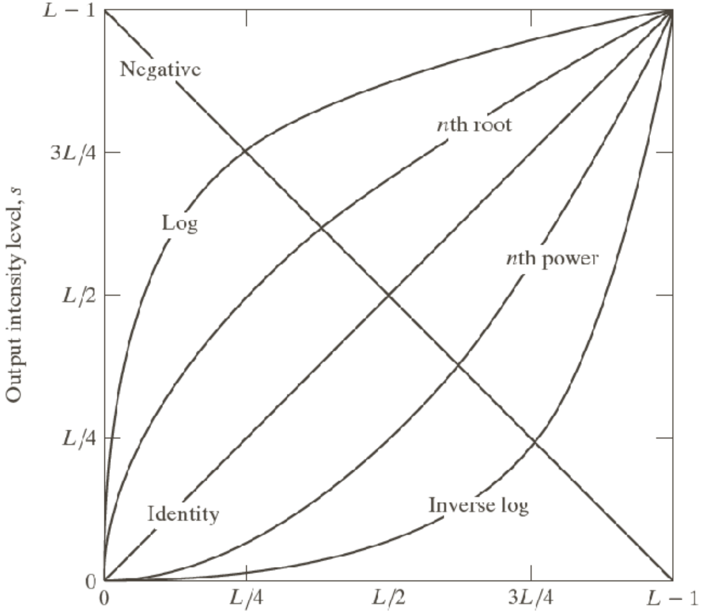
$$s = c \log(1 + r)$$



a b

FIGURE 3.5 (a) Fourier spectrum. (b) Result of applying the log transformation in Eq. (3.2-2) with $c = 1$.

Transformation Function



Gamma Transformation Function

$$s = cr^{\gamma}$$

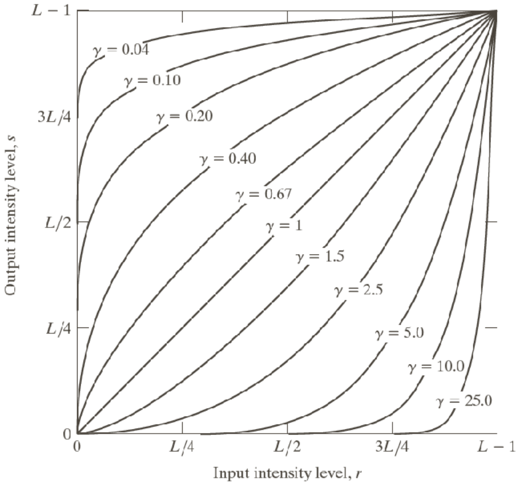
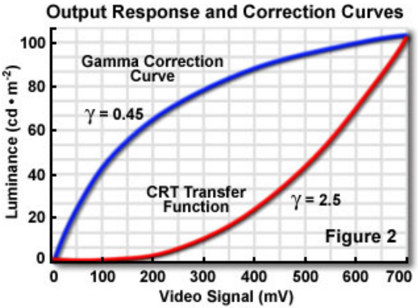
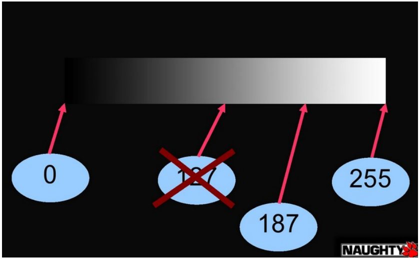


FIGURE 3.6 Plots of the equation $s = cr^{\gamma}$ for various values of γ ($c = 1$ in all cases). All curves were scaled to fit in the range shown.

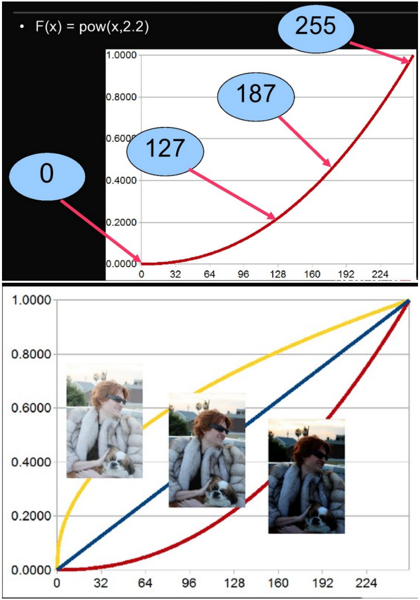


Output Response and Correction Curves
Luminance (cd • m^-2)
100
80
60
40
20
0
Gamma Correction Curve
γ = 0.45
CRT Transfer Function
γ = 2.5
Figure 2
0
100
200
300
400
500
600
700
Video Signal (mV)



0
255
187

Gamma Transformation Function



Gamma Transformation Function

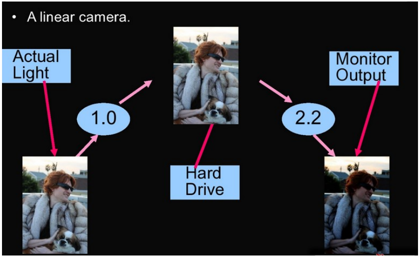
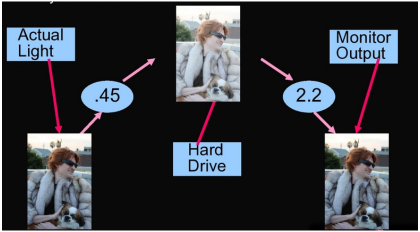
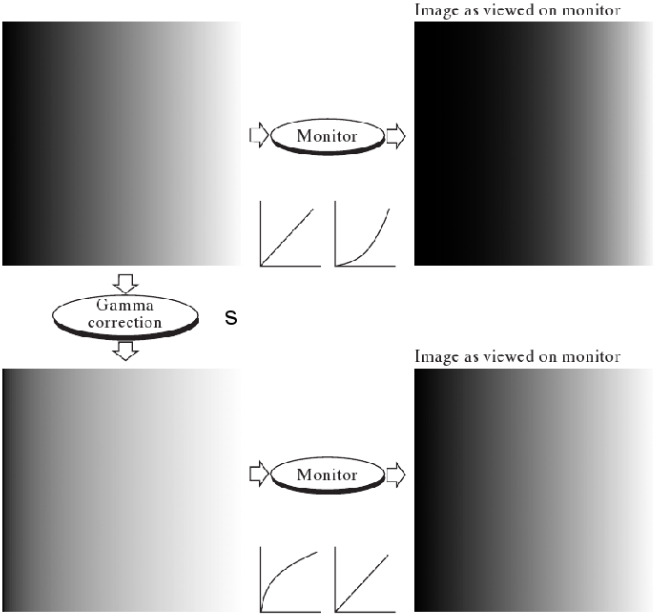


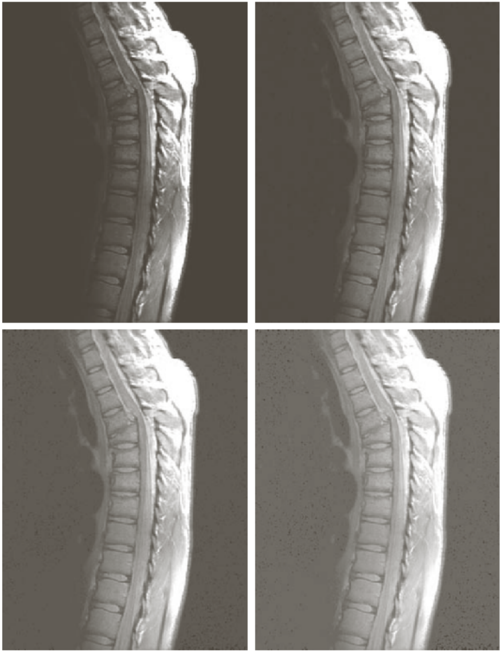
Image is corrected before visualization



a b
c d

FIGURE 3.7
(a) Linear-wedge gray-scale image.
(b) Response of monitor to linear wedge.
(c) Gamma-corrected wedge.
(d) Output of monitor.





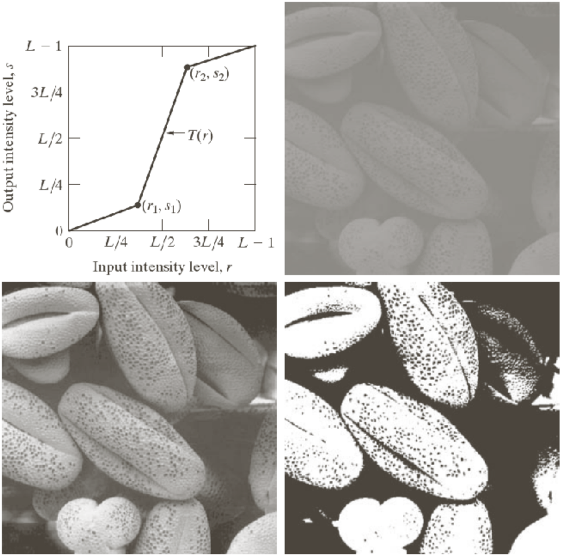
a b
c d

FIGURE 3.8 (a) Magnetic resonance image (MRI) of a fractured human spine. (b)–(d) Results of applying the transformation in Eq. (3.2-3) with $c - 1$ and $\gamma = 0.6, 0.4$, and 0.3, respectively. (Original image courtesy of Dr. David R. Pickens, Department of Radiology and Radiological Sciences, Vanderbilt University Medical Center.)

a	b
c	d

FIGURE 3.9 (a) Aerial image. (b)–(d) Results of applying the transformation in Eq. (3.2-3) with $c = 1$ and $\gamma = 3.0, 4.0$, and 5.0, respectively. (Original image for this example courtesy of NASA.)

Piecewise-Linear Transformation Fucntions

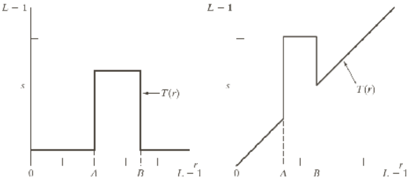


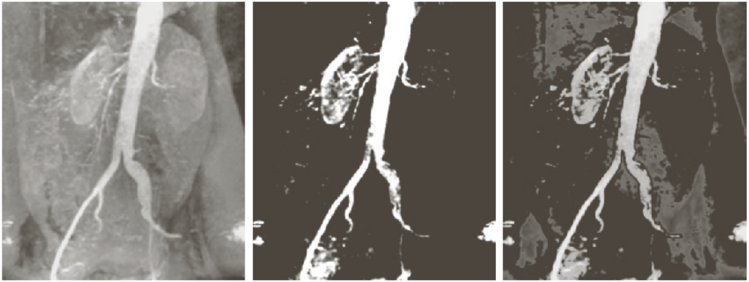
a b
c d

FIGURE 3.10 Contrast stretching. (a) Form of transformation function. (b) A low-contrast image. (c) Result of contrast stretching. (d) Result of thresholding. (Original image courtesy of Dr. Roger Heady, Research School of Biological Sciences, Australian National University, Canberra, Australia.)

a b

FIGURE 3.11 (a) This transformation highlights intensity range $[A, B]$ and reduces all other intensities to a lower level. (b) This transformation highlights range $[A, B]$ and preserves all other intensity levels.

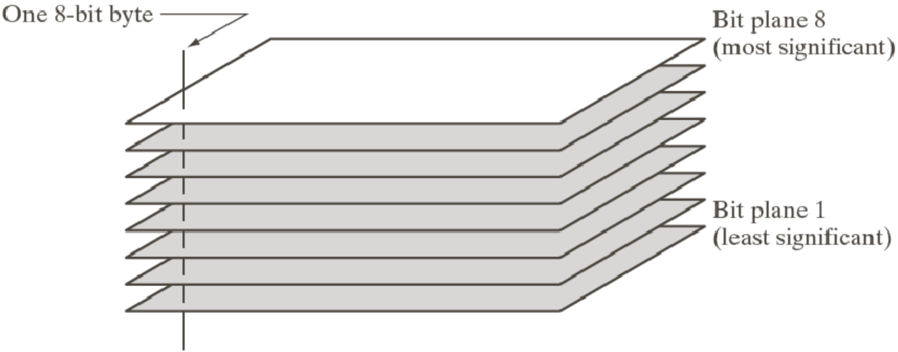




a b c

FIGURE 3.12 (a) Aortic angiogram. (b) Result of using a slicing transformation of the type illustrated in Fig. 3.11(a), with the range of intensities of interest selected in the upper end of the gray scale. (c) Result of using the transformation in Fig. 3.11(b), with the selected area set to black, so that grays in the area of the blood vessels and kidneys were preserved. (Original image courtesy of Dr. Thomas R. Gest, University of Michigan Medical School.)

Bit-plane slicing



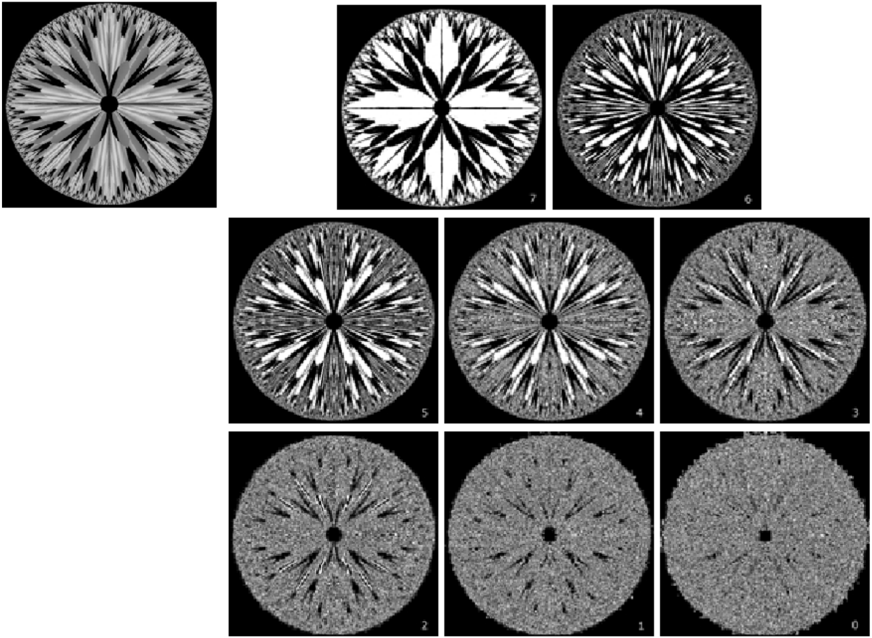


FIGURE 3.14 The eight bit planes of the image in Fig. 3.13. The number at the bottom, right of each image identifies the bit plane.

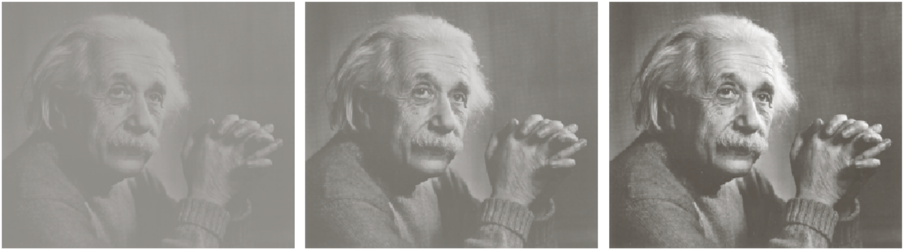
a b c
d e f
g h i

FIGURE 3.14 (a) An 8-bit gray-scale image of size 500 × 1192 pixels. (b) through (i) Bit planes 1 through 8, with bit plane 1 corresponding to the least significant bit. Each bit plane is a binary image.

a b c

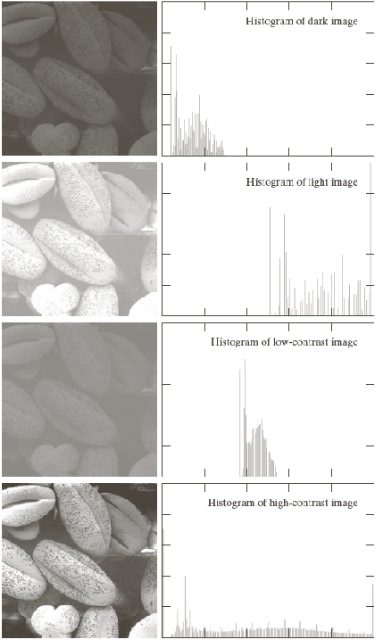
FIGURE 3.15 Images reconstructed using (a) bit planes 8 and 7; (b) bit planes 8, 7, and 6; and (c) bit planes 8, 7, 6, and 5. Compare (c) with Fig. 3.14(a).

Contrast



a b c

FIGURE 2.41 Images exhibiting (a) low contrast, (b) medium contrast, and (c) high contrast.



- Discrete Function:

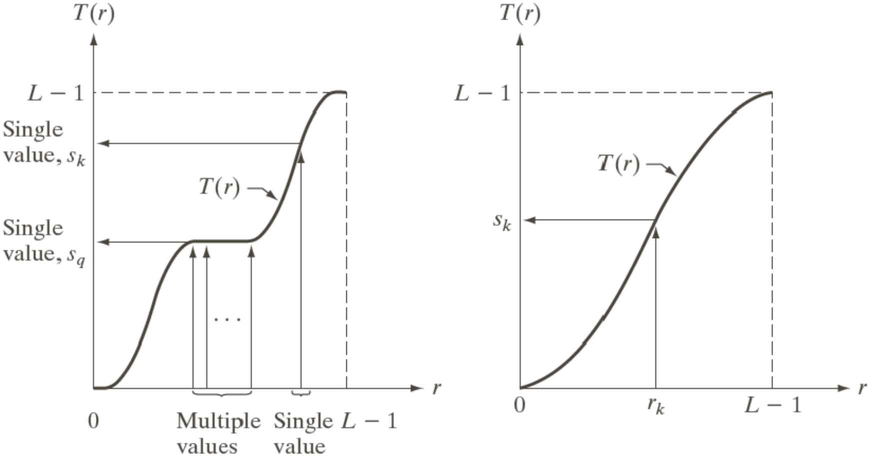
$$h(r_k) = n_k$$

  - r_k – intensity level
  - n_k – number of pixels with intensity level equal to r_k
- Normalized
  - total number of pixels
  - probability
  - sum $= 1$

Histogram Transformation

- The histogram transformation function – transforms a pixel distribution into a more 'interesting' distribution;
- A function T(r) is used, which must satisfy the following conditions:
  1. Be monotonically increasing in the interval $0 \leq r \leq (L-1)$
  2. $0 \leq T(r) \leq L-1$ for $0 \leq r \leq (L-1)$

Histogram Transformation



Histogram Transformation

- p_r e p_s are probability density functions (PDF);
- $T(r)$ is the function that transforms a variable (r) into another (s);

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- If $pr(r)$ and $T(r)$ are known, with $T(r)$ continuous and differentiable, $p_s(s)$ can be obtained by:

$$p_s(s) = p_r(r)\left|\frac{dr}{ds}\right|$$

Histogram Transformation

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$$p_s(s) = p_r(r) \left| \frac{dr}{ds} \right|$$

- One example of $T(r)$:

$$s = T(r) = (L-1) \int_0^r p_r(w) dw$$

Considering:

$$s = T(r) = (L-1)\int_0^r p_r(w)dw$$

Histogram Transformation

Considering:

$$s = T(r) = (L-1)\int_0^r p_r(w)dw$$

$$\frac{ds}{dr} = \frac{dT(r)}{dr}$$

Histogram Transformation

Considering:

$$s = T(r) = (L-1)\int_0^r p_r(w)dw$$

$$\frac{ds}{dr} = \frac{dT(r)}{dr}$$

$$\frac{ds}{dr} = (L-1)\frac{d}{dr}\int_0^r p_r(w)dw = (L-1)p_r(r)$$

Histogram Transformation

Considering:

$$s = T(r) = (L-1)\int_0^r p_r(w)dw$$

$$\frac{ds}{dr} = \frac{dT(r)}{dr}$$

$$\frac{ds}{dr} = (L-1)\frac{d}{dr}\int_0^r p_r(w)dw = (L-1)p_r(r)$$

$$p_s(s) = p_r(r)\left|\frac{dr}{ds}\right| = p_r(r)\left|\frac{1}{p_r(r)(L-1)}\right| = \frac{1}{L-1}$$

Uniform Distribution

$$s = T(r) = (L-1)\int_0^r p_r(w)dw$$

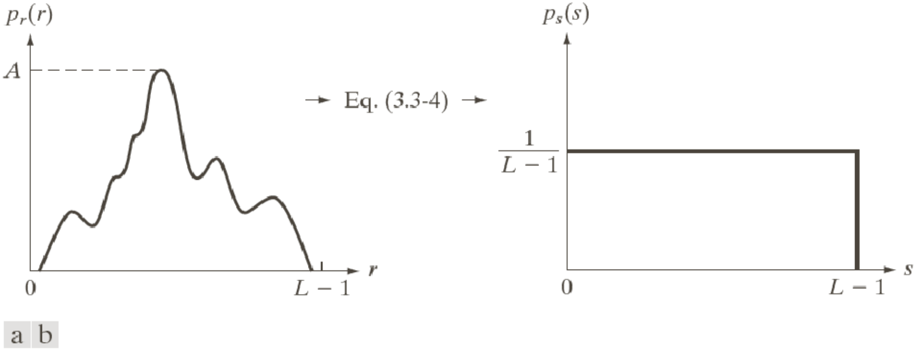

FIGURE 3.18 (a) An arbitrary PDF. (b) Result of applying the transformation in Eq. (3.3-4) to all intensity levels, r. The resulting intensities, s, have a uniform PDF, independently of the form of the PDF of the r's.

Histogram Transformation

- Continuous Case:

$$s = T(r) = (L-1)\int_0^r p_r(w)dw$$

Histogram Transformation

- Continuous Case:

$$s = T(r) = (L-1)\int_0^r p_r(w)dw$$

- Discrete Case:

For $k = 0, 1, \ldots, L-1$.

$$\begin{aligned} s_k = T(r_k) &= (L-1)\sum_{j=0}^{k} p_r(r_j)s_k \\ &= (L-1)\sum_{j=0}^{k} \frac{n_j}{M\cdot N} \\ &= \frac{(L-1)}{M\cdot N}\sum_{j=0}^{k} n_j \end{aligned}$$

Ex: 3 bits Image ($L = 8$) of size 64× 64 pixels ($M \cdot N = 4096$)

r_k	n_k	$p_r(r_k) = n_k/MN$
$r_0 = 0$	790	0.19
$r_1 = 1$	1023	0.25
$r_2 = 2$	850	0.21
$r_3 = 3$	656	0.16
$r_4 = 4$	329	0.08
$r_5 = 5$	245	0.06
$r_6 = 6$	122	0.03
$r_7 = 7$	81	0.02

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$$s_0 = T(r_0) = 7 \sum_{0}^{0} p_r(r_j) = 7 \cdot 0,19 = 1,33 \longrightarrow 1$$

$$s_1 = T(r_1) = 7 \sum_{0}^{1} p_r(r_j) = 7 \cdot (0,19 + 0,25) = 3,08 \longrightarrow 3$$

$$s_2 = 4,55 \longrightarrow 5 \quad s_3 = 5,67 \longrightarrow 6$$
$$s_4 = 6,23 \longrightarrow 6 \quad s_5 = 6,65 \longrightarrow 7$$
$$s_6 = 6,86 \longrightarrow 7 \quad s_7 = 7,00 \longrightarrow 7$$

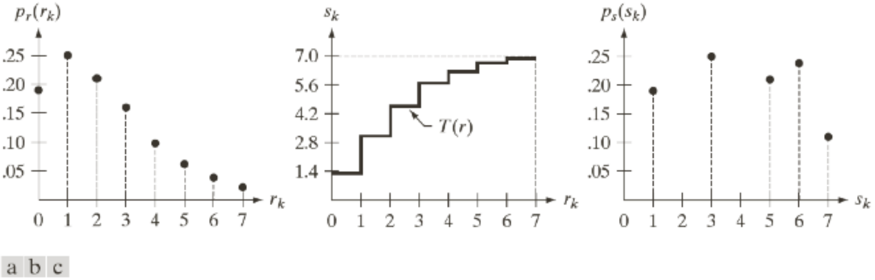

FIGURE 3.19 Illustration of histogram equalization of a 3-bit (8 intensity levels) image. (a) Original histogram. (b) Transformation function. (c) Equalized histogram.

$$s_k = T(r_k)$$

$$= \frac{(L-1)}{M \cdot N} \sum_{j=0}^{k} n_j$$

r_k	n_k	$p_r(r_k) = n_k/MN$
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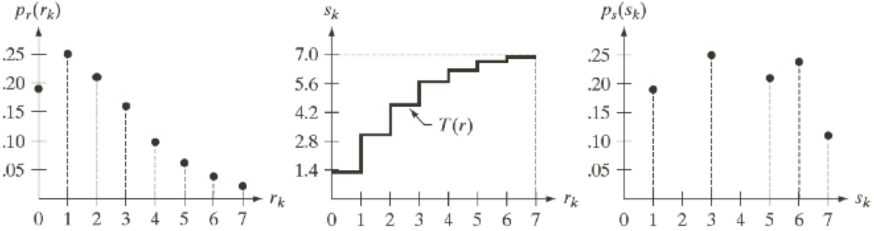

a b c

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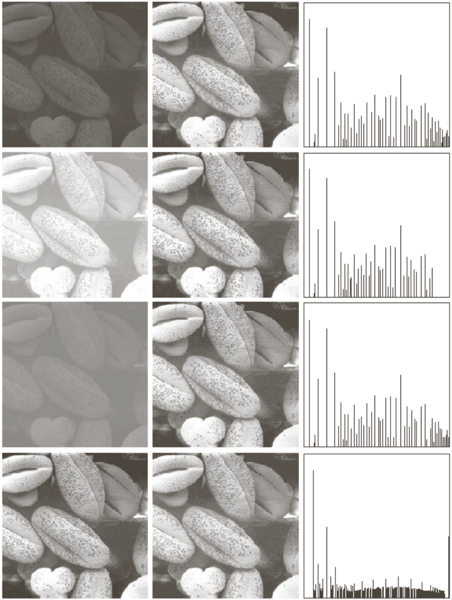
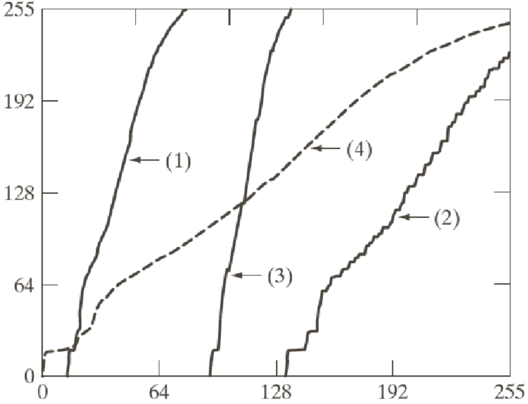
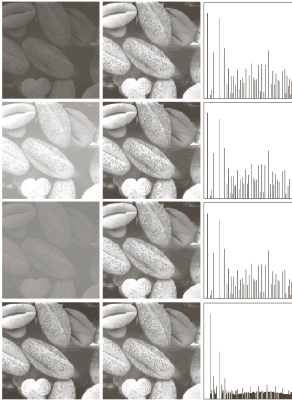
Histogram Transformation



Histogram Transformation





Histogram Transformation

Question:

Does the histogram equalization always gives good results?

Question:

Does the histogram equalization always gives good results? NO!

- An image has a specific histogram;
- Let $p_r(r)$ and $p_z(z)$ be the probability density functions of the variables r e z.
- $p_z(z)$ is the *specified* probability density function;
- Let s be a random variable

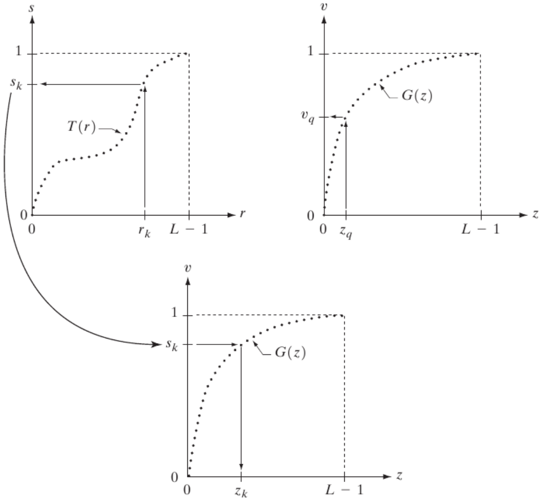
$$s = T(r) = (L-1)\int_0^r p_r(w)dw$$

Histogram Specification

- We obtain the function G:

$$s = T(r) = (L-1)\int_0^r p_r(w)dw$$
$$G(z) = (L-1)\int_0^z p_z(t)dt = s$$
$$z = G^{-1}(s) = G^{-1}[T(r)]$$

Histogram Specification



Histogram Specification

1. Obtain the $p_r(r)$ of the input image and calculate the s values:

$$s = T(r) = (L-1)\int_0^r p_r(w)dw$$

2. Use the specified PDF to obtain the function $G(z)$:

$$G(z) = (L-1)\int_0^z p_z(t)dt = s$$

3. Map s to z

$$z = G^{-1}(s)$$

- **Ex.:** If we assume continuous values:

$$p_r(r) = \begin{cases} \frac{2r}{(L-1)^2}, & \text{para} 0 \leq r \leq L-1 \\ 0, & \text{caso contrário} \end{cases} \quad (1)$$

- Find the transformations thah produces:

$$p_z(z) = \begin{cases} \frac{3z^2}{(L-1)^3}, & \text{para} 0 \leq z \leq L-1 \\ 0, & \text{caso contrário} \end{cases} \quad (2)$$

Histogram Specification

Steps:

- Find the equalization transformation:

$$
\begin{aligned}
s = T(r) &= (L-1)\int_0^r p_r(w)dw \\
&= (L-1)\int_0^r \frac{2w}{(L-1)^2}dw = \frac{r^2}{L-1}
\end{aligned}
$$

Steps:

- Find the equalization transformation:

$$s = T(r) = (L-1)\int_0^r p_r(w)dw$$
$$= (L-1)\int_0^r \frac{2w}{(L-1)^2}dw = \frac{r^2}{L-1}$$

- The transformation for the histogram:

$$G(z) = (L-1)\int_0^z p_z(t)dt$$
$$= (L-1)\int_0^z \frac{3t^2}{(L-1)^3}dt = \frac{z^3}{(L-1)^2}$$

- Transformation Function:

$$z = \left[(L-1)^2 s\right]^{1/3} = \left[(L-1)^2 \frac{r^2}{L-1}\right]^{1/3}$$
$$= \left[(L-1) r^2\right]^{1/3}$$

- So, to obtain the transformed image, we apply the above function in the r variable.

Discrete case:

- Obtain $p_r(r_j)$ of the input image and, later, the s_k values, rounding the values to integers in the interval $[0, L-1]$:

$$s_k = T(r_k) = \frac{(L-1)}{M \cdot N} \sum_{j=0}^{k} n_j$$

- Use the specified PDF to obtain the function $G(z_q)$, rounding the values to integers in the interval $[0, L-1]$:

$$G(z_k) = (L-1) \sum_{i=0}^{q} p_z(z_i) = s_k$$

- Map values of s_k to z_q:

$$z_q = G^{-1}(s_k)$$

- In practice, we do not calculate the inverse ...
- We calculate all possible values of G to every q

$$G(z_k) = (L-1)\sum_{i=0}^{q} p_z(z_i) = s_k$$

- This values are scaled and rounded to the nearest integers in the interval $[0, L-1]$ and saved (table);
- For a given value s_k, we find the closest value in this table.

Histogram Specification

Discrete Case:

1. Calculate the histogram of the input image, $p_r(r)$, and use it to calculate the equalization transformation:

$$s_k = T(r_k) = \frac{(L-1)}{M \cdot N} \sum_{j=0}^{k} n_j$$

   Round the results, s_k, to integers in the interval $[0, L-1$

2. Calculate all transformation values G for $q = 0, 1, \ldots, L-1$

$$G(z_k) = (L-1) \sum_{i=0}^{q} p_z(z_i) = s_k$$

   Round the results of G to integers in the interval $[0, L-1$ e save these results in a table.

Discrete Case:

1. For each value of s_k, $k = 0, 1, \dots, L-1$, use the values of G (step 2) to find the corresponding values of z_q, in a way that $G(z_q)$ is the closest to s_k. Save this mapping values form s to z.
2. Build an image with a specified histogram, performing first the equalization and, then, the mapping from s_k to z_q.

Example:

- 3 bits image ($L = 8$), 64×64 ($M \cdot N = 4096$) pixels

r_k	n_k	$p_r(r_k) = n_k/MN$
$r_0 = 0$	790	0.19
$r_1 = 1$	1023	0.25
$r_2 = 2$	850	0.21
$r_3 = 3$	656	0.16
$r_4 = 4$	329	0.08
$r_5 = 5$	245	0.06
$r_6 = 6$	122	0.03
$r_7 = 7$	81	0.02

z_q	Specified $p_z(z_q)$	Actual $p_z(z_k)$
$z_0 = 0$	0.00	0.00
$z_1 = 1$	0.00	0.00
$z_2 = 2$	0.00	0.00
$z_3 = 3$	0.15	0.19
$z_4 = 4$	0.20	0.25
$z_5 = 5$	0.30	0.21
$z_6 = 6$	0.20	0.24
$z_7 = 7$	0.15	0.11

Histogram Specification

Example:

r_k	n_k	$p_r(r_k) = n_k/MN$
$r_0 = 0$	790	0.19
$r_1 = 1$	1023	0.25
$r_2 = 2$	850	0.21
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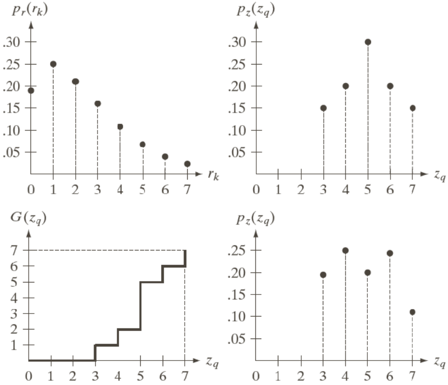
$z_4 = 4$	0.20	0.25
$z_5 = 5$	0.30	0.21
$z_6 = 6$	0.20	0.24
$z_7 = 7$	0.15	0.11



Exemplo:

- Obtain equalized samples:

$$\begin{array}{ll} s_0 = 1,33 \longrightarrow 1 & s_1 = 3,08 \longrightarrow 3 \\ s_2 = 4,55 \longrightarrow 5 & s_3 = 5,67 \longrightarrow 6 \\ s_4 = 6,23 \longrightarrow 6 & s_5 = 6,65 \longrightarrow 7 \\ s_6 = 6,86 \longrightarrow 7 & s_7 = 7,00 \longrightarrow 7 \end{array}$$

- Calculate the transformation value:

$$G(z_0) = 7 \sum_{j=0}^{0} p_z(z_j) = 0,00 \longrightarrow 0$$

$$\begin{array}{ll} G(z_1) = 0,00 & G(z_2) = 0,00 \\ G(z_3) = 1,05 & G(z_4) = 2,45 \\ G(z_5) = 4,55 & G(z_6) = 5,95 \\ G(z_7) = 7,00 & \end{array}$$

Example:

- Obtain equalized samples:

$$\begin{array}{ll} s_0 = 1,33 \longrightarrow 1 & s_1 = 3,08 \longrightarrow 3 \\ s_2 = 4,55 \longrightarrow 5 & s_3 = 5,67 \longrightarrow 6 \\ s_4 = 6,23 \longrightarrow 6 & s_5 = 6,65 \longrightarrow 7 \\ s_6 = 6,86 \longrightarrow 7 & s_7 = 7,00 \longrightarrow 7 \end{array}$$

- Calculate the transformation values and **round it to the closest integers**:

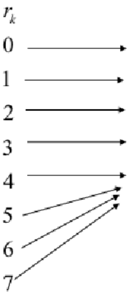
$$G(z_0) = 7\sum_{j=0}^{0} p_z(z_j) = 0,00 \longrightarrow 0$$

$$\begin{array}{ll} G(z_1) = 0,00 \longrightarrow 0 & G(z_2) = 0,00 \longrightarrow 0 \\ G(z_3) = 1,05 \longrightarrow 1 & G(z_4) = 2,45 \longrightarrow 2 \\ G(z_5) = 4,55 \longrightarrow 5 & G(z_6) = 5,95 \longrightarrow 6 \\ G(z_7) = 7,00 \longrightarrow 7 & \end{array}$$

Histogram Specification

$$s_0 = 1, s_1 = 3, s_2 = 5, s_3 = 6, s_4 = 7,$$
$$s_5 = 7, s_6 = 7, s_7 = 7.$$

z_q	$G(z_q)$
$z_0 = 0$	0
$z_1 = 1$	0
$z_2 = 2$	0
$z_3 = 3$	1
$z_4 = 4$	2
$z_5 = 5$	5
$z_6 = 6$	6
$z_7 = 7$	7



s_k	$\rightarrow$	z_q
1	$\rightarrow$	3
3	$\rightarrow$	4
5	$\rightarrow$	5
6	$\rightarrow$	6
7	$\rightarrow$	7

Find the smallest value of z_q in a way that $G(z_q)$ is the closest value to s_k.

Histogram Specification

$$r_k \rightarrow z_q$$

$$0 \rightarrow 3$$

$$1 \rightarrow 4$$

$$2 \rightarrow 5$$

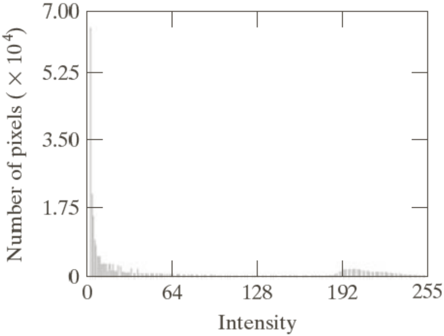
$$3 \rightarrow 6$$

$$4 \rightarrow 7$$

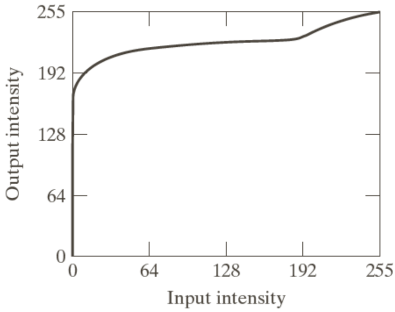
$$5 \rightarrow 7$$

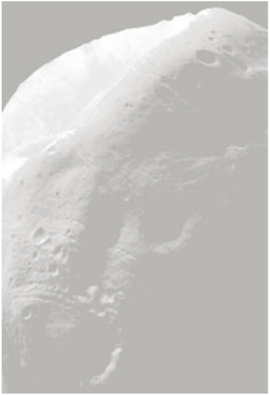
$$6 \rightarrow 7$$

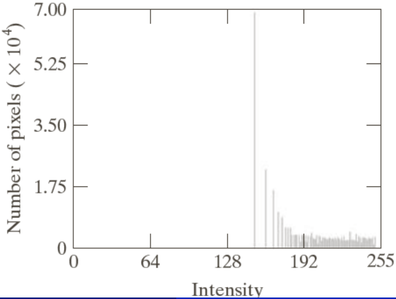
$$7 \rightarrow 7$$

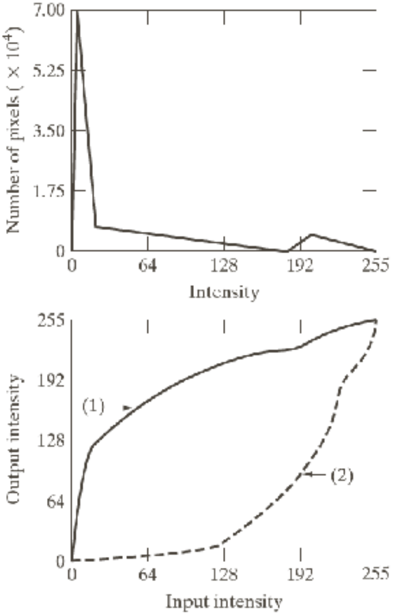

7.00
5.25
3.50
1.75
0
Number of pixels (× 10^4)
0
64
128
192
255
Intensity

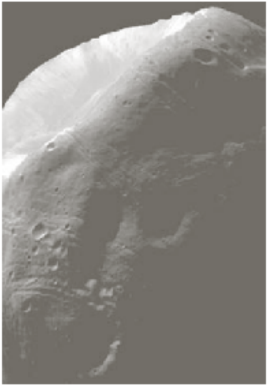
Equalized


Output intensity
Input intensity
0
64
128
192
255




Number of pixels (× 10^4)
Intensity
0
1.75
3.50
5.25
7.00
64
128
192
255

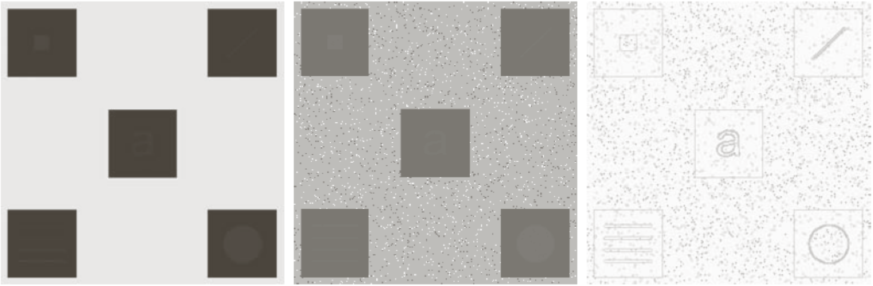

7.00
5.25
3.50
1.75
0
Number of pixels (× 10⁴)
0
64
128
192
255
Intensity
255
192
128
64
0
Output intensity
(1)
(2)
0
64
128
192
255
Input intensity



Local Processing of Histograms

- Define a neighborhood and move the the central position pixel by pixel;
- In each position, the histograms of the neighborhood is calculated. We can perform a histogram equalization or a histogram specification;
- Map the intensity of the central neighborhood pixel;
- Move to the next positions and repeat the procedure.

Local Processing of Histograms



a b c

FIGURE 3.26 (a) Original image. (b) Result of global histogram equalization. (c) Result of local histogram equalization applied to (a), using a neighborhood of size 3×3.

Histogram Statistics

- Intensity average

$$m = \sum_{i=0}^{L-1} r_i p(r_i) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y)$$

- Intensity Variance

$$\sigma^2 = u_2(r) = \sum_{i=0}^{L-1} (r_i - m)^2 p(r_i) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} [f(x, y) - m]^2$$

- n-th Moment

$$u_n(r) = \sum_{i=0}^{L-1} (r_i - m)^n p(r_i)$$

- Local Intensity Average:

$$m_{S_{xy}} = \sum_{i=0}^{L-1} r_i p_{S_{xy}}(r_i)$$

- Local Intensity Variance

$$\sigma^2_{S_{xy}} = \sum_{i=0}^{L-1} (r_i - m_{S_{xy}})^2 p_{S_{xy}}(r_i)$$

S_{xy} – neighborhood

Local Histogram Processing

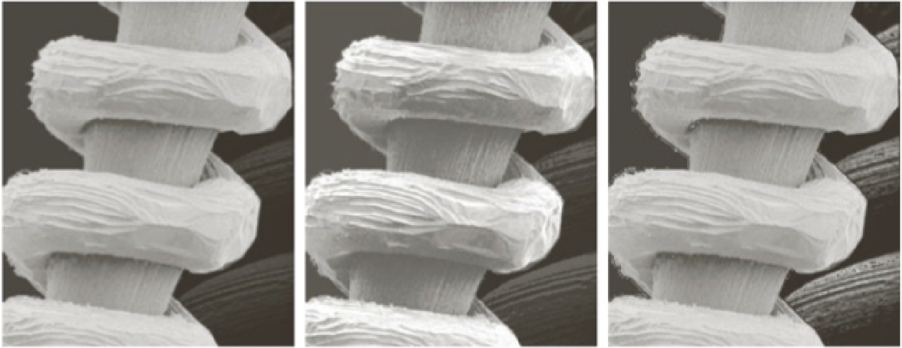
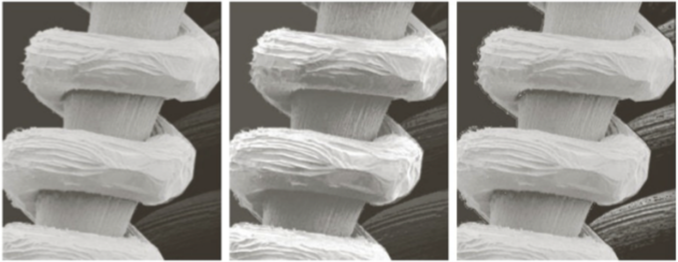


FIGURE 3.27 (a) SEM image of a tungsten filament magnified approximately 130×. (b) Result of global histogram equalization. (c) Image enhanced using local histogram statistics. (Original image courtesy of Mr. Michael Shaffer, Department of Geological Sciences, University of Oregon, Eugene.)

Local Histogram Processing



a b c

FIGURE 3.27 (a) SEM image of a tungsten filament magnified approximately 130×. (b) Result of global histogram equalization. (c) Image enhanced using local histogram statistics. (Original image courtesy of Mr. Michael Shaffer, Department of Geological Sciences, University of Oregon, Eugene.)

$$g(x,y) = \begin{cases} E g f(x,y), & \text{if } m_{s_{xy}} \le k_0 m_G \text{ and } k_1 \sigma_G \le \sigma_{s_{xy}} \le k_2 \sigma_G \\ f(x,y), & \text{otherwise} \end{cases}$$

m_G : global mean; σ_G : global standard deviation

$k_0 = 0.4$; $k_1 = 0.02$; $k_2 = 0.4$; $E = 4$